Elastic Properties and Prime Elements

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Abstract. In a commutative, cancellative, atomic monoid M, the elasticity of a non-unit x is defined to be $\rho(x) = L(x)/l(x)$, where $L(x)$ is the supremum of the lengths of factorizations of x into irreducibles and $l(x)$ is the corresponding infimum. The elasticity $\rho(M)$ of M is given as the supremum of the elasticities of the nonzero non-units in the domain. We call $\rho(M)$ accepted if there exists a non-unit $x \in M$ with $\rho(M) = \rho(x)$. In this paper, we show for a monoid M with accepted elasticity that

$$\{\,\rho(x) \mid x \text{ a non-unit of } M\} = \mathbb{Q} \cap [1, \rho(M)]$$

if M has a prime element. We develop the ideas of taut and flexible elements to study the set $\{\,\rho(x) \mid x \text{ a non-unit of } M\}$ when M does not possess a prime element.

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1. Introduction and Preliminaries

Let M be a commutative, cancellative monoid, $M^\bullet$ its set of non-units and $M^\times$ its set of units. Recall that $x \in M^\bullet$ is *irreducible* if $x = yz$ in M implies that either y or z is in $M^\times$. Let $\mathcal{A}(M)$ represent the set of irreducible elements (or atoms) of M. If every $x \in M^\bullet$ can be written as a product of elements from $\mathcal{A}(M)$, then call M *atomic*, and we assume throughout the remainder of our work that M has this property. An element $x \in \mathcal{A}(M)$ is *prime* if whenever $x \mid yz$ in M then either $x \mid y$ or $x \mid z$. Since factorization in $M^\bullet$ into irreducible elements is

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not necessarily unique, the question of the possible lengths of factorizations into irreducibles arises. Hence, for $x \in M^\bullet$, define

$$\mathcal{L}(x) = \{n \mid x = \alpha_1 \cdots \alpha_n \text{ with each } \alpha_i \in \mathcal{A}(M)\}$$

to be the *set of lengths* of factorizations of x into atoms and

$$\mathfrak{L}(M) = \{\, \mathcal{L}(x) \;\mid\; x \in M^\bullet \,\}$$

to be the *set of lengths of* M. In addition, define

$$L(x) = \sup \mathcal{L}(x),\; l(x) = \inf \mathcal{L}(x), \;\text{ and }\; \rho(x) = \frac{L(x)}{l(x)}.$$

If $L(x) < \infty$ for each $x \in M^\bullet$, then M is called a *bounded factorization monoid* (or *BFM*). The invariant $\rho(x)$ is called the *elasticity* of x and hence if M is a BFM then each $x \in M^\bullet$ has rational elasticity. We further define

$$\mathcal{R}(M) = \{\, \rho(x) \mid x \in M^\bullet\}$$

as the set of elasticities of the elements in $M^\bullet$. The elasticity is defined globally for M as $\rho(M) = \sup \mathcal{R}(M)$. We call $\rho(M)$ *accepted* if there exists an $x \in M^\bullet$ with $\rho(M) = \rho(x)$ (by [1], any finitely generated M has accepted elasticity). We define M to be *fully elastic* if $\mathcal{R}(M) \supseteq \mathbb{Q} \cap [1, \rho(M))$. Chapman, Holden, and Moore showed in [9] that numerical monoids are fully elastic if and only if they are cyclic. Their paper also showed that a ring of algebraic integers is fully elastic if its class number is a prime power. Full elasticity has also been explored for rings of integer-valued polynomials in [11]. In light of the results of [6], we have modified the definition of fully elastic given in [9] so that monoids without accepted elasticity can satisfy this condition (see Example 2.12).

In this note, we show in Section 2 that an atomic monoid with accepted elasticity is fully elastic if M possesses a prime element. Even if M does not have accepted elasticity, we exhibit an additional property (see Definition 1.1 below) that guarantees that M is fully elastic when possessing a prime element. We illustrate these results with numerous examples and show that they can fail if M does not have a prime element. In Section 3, we consider the case where M does not contain a prime element. With the development of *flexible* elements, we are able to give in Theorems 3.4 and 3.6 criteria for such an M to be fully elastic. We close in Section 4 by extending an example given in [9] and show that all power series rings of the form $K[[x^{n_1}, x^{n_2}, \ldots, x^{n_t}]]$ where K is any field, $S = \langle n_1, \ldots, n_t\rangle$ is a primitive numerical monoid and $t \geq 2$, are not fully elastic. For readers unfamiliar with the theory of non-unique factorizations, a good general reference is [13].

A notion related to full elasticity was recently introduced and explored in [6]. Define the *asymptotic elasticity* of $x \in M^\bullet$ to be

$$\overline{\rho}(x) = \lim_{n\to\infty} \rho(x^n).$$

Let $\overline{\mathcal{R}}(M) = \{\overline{\rho}(x) \mid x \in M^\bullet\}$ and $\overline{\rho}(M) = \sup \overline{\mathcal{R}}(M)$. M is called *asymptotically fully elastic* if $\overline{\mathcal{R}}(M) \supseteq \mathbb{Q} \cap [1, \overline{\rho}(M))$.

In studying problems involving full or asymptotic elasticity, the notion of a taut element plays in important role.

Definition 1.1. Let M be an atomic monoid. An element $f \in M^\bullet$ is *taut* if there exists $m \in \mathbb{N} \cup \{0\}$ such that $\rho(f^m) = \overline{\rho}(f)$. We call M *taut* if every $f \in M^\bullet$ is taut.

By [1, Theorem 12] (or [13, 3.8.1]) an atomic monoid M is taut provided that the set

$$\mathcal{U}(y) = \{x \in \mathcal{A}(M) \mid \exists n \in \mathbb{N}\ x|y^n\}$$

contains finitely many non-associated irreducibles for every $y \in M^\bullet$. It is not known if this is a necessary condition. It follows that if M is locally finitely generated, then it is taut. Hence, numerical monoids and block monoids over abelian torsion groups are taut. By [6, Corollary 6], an asymptotically fully elastic M which is taut must also be fully elastic. In [6], a taut monoid M is called a $\mathcal{U}$-monoid. We prefer the present, more descriptive name.

2. General Results on Full and Asymptotic Elasticity

If M is a BFM, then the taut hypothesis implies that $\overline{\rho}(x) < \infty$ for every $x \in M^\bullet$. If M is not taut, then this may not be the case. For example, in $M = \mathbb{Z}[\sqrt{-7}]$, $\overline{l}(2) = 0$ so $\overline{\rho}(2) = \infty$ [12, Example 11]. We open with a crucial lemma.

Lemma 2.1. *Let M be an atomic monoid containing a prime element p and an element f such that $\rho(f) = \overline{\rho}(f)$. Then for each element $\alpha \in \mathbb{Q} \cap [1, \rho(f)]$, there exists $g \in M$ such that $\rho(g) = \overline{\rho}(g) = \alpha$.*

Proof. First we notice that for any positive integer i, $\rho(f^i) \geq \rho(f)$. This follows since $\mathcal{L}(f^i)$ must contain the elements $iL(f)$ and $il(f)$. Hence $L(f^i) \geq iL(f)$ and $l(f^i) \leq il(f)$, and so $\rho(f^i) \geq \rho(f)$. In addition, by the definition of $\overline{\rho}(f)$, it must be the case that $\rho(f^i) \leq \overline{\rho}(f)$. Since $\rho(f^i) \geq \rho(f)$ and $\rho(f^i) \leq \overline{\rho}(f) = \rho(f)$, then we see that $\rho(f^i) = \rho(f)$. Since this holds for any integer $i > 0$, $\rho(f^i) = \rho(f)$ and it follows that $L(f^i) = iL(f)$ and $l(f^i) = il(f)$.

Now let $a/b \in [1, \rho(f)] \cap \mathbb{Q}$ be given. Let $i = a - b$ and $j = L(f)b - l(f)a$. Note that these are both nonnegative if $1 \leq a/b \leq \rho(f)$. Consider the element $h = f^i p^j$. Since p is a prime element in M, it must be contained in every factorization of h into irreducibles of M. Therefore we have (from the results above) that $L(h) = iL(f) + j$ and $l(h) = il(f) + j$, so

$$\begin{aligned}\rho(h) &= \frac{iL(f)+j}{il(f)+j}\\ &= \frac{(a-b)L(f)+L(f)b-l(f)a}{(a-b)l(f)+L(f)b-l(f)a}\\ &= \frac{a\,(L(f)-l(f))}{b\,(L(f)-l(f))}\\ &= a/b,\end{aligned}$$

and a/b is the elasticity (and the asymptotic elasticity) of an element of M. □

Corollary 2.2. *Let M be an atomic monoid with accepted elasticity. If M contains a prime element, then M is fully elastic and asymptotically fully elastic.*

Proof. If M has accepted elasticity, there is an element $f \in M$ such that $\rho(f) = \rho(M)$. Since f has maximal elasticity, it follows from [6, Lemma 1] that $\rho(f) = \overline{\rho}(f)$. Now we may apply Lemma 2.1 to f, completing the proof. □

We are able to apply Corollary 2.2 to a wide variety of examples. Before doing so we will require two results concerning classical extensions of commutative rings.

Lemma 2.3. *Let $\overline{R}$ be a domain and R an order in $\overline{R}$ (that is, $R \subseteq \overline{R}$ is a subring with the same quotient field, and $\overline{R}$ is a finitely generated R-module). Let $F = (R : \overline{R})$ be the conductor of R in $\overline{R}$ and $p \in R$ such that $p + F \in (R/F)^\times$. Then p is a prime element of R if and only if it is a prime element of $\overline{R}$.*

Proof. Let $f \in F$ and $q \in R$ be such that $pq + f = 1$. Let first p be a prime element of $\overline{R}$ and $a, b \in R$ such that p divides ab in R. Then p divides ab in $\overline{R}$, and we may assume that p divides a, say $a = pc$ for some $c \in \overline{R}$. Then $c + F = (a + F)(p + F)^{-1} \in R/F$ and thus $c \in R + F = R$. Hence p divides a in R.

Now, let p be a prime element of R and $a, b \in \overline{R}$ such that p divides ab, say $ab = pc$ for some $c \in \overline{R}$. Then $(af)(bf) = p(cf^2)$ shows that p divides $(af)(bf)$ in R, and we may assume that p divides af. Then p divides $a = af + apq$. □

Theorem 2.4. *Let $\overline{R}$ be a Dedekind domain, R an order in $\overline{R}$, $F = (R : \overline{R})$, and assume that $\mathrm{Pic}(\overline{R})$ and $(\overline{R}/F)^\times$ are finite. Then $\mathrm{Pic}(R)$ is finite, and if for every prime ideal P of R containing F there is exactly one prime ideal of $\overline{R}$ lying above P, then R is taut and has accepted elasticity. Further, if $\overline{R}$ is the ring of integers of an algebraic number field or a holomorphy ring in an algebraic function field for a finite set of places, then R contains a prime element.*

Proof. The finiteness of $\mathrm{Pic}(R)$ follows from the exact sequence

$$(R/F)^\times \to \mathrm{Pic}(R) \to \mathrm{Pic}(\overline{R}) \to 0$$

(see [19, §12]). Assume now that for every prime ideal P of R containing F there is exactly one prime ideal of $\overline{R}$ lying above P. Then R has accepted elasticity by

[15, Theorem 5], and by [13, Theorem 3.7.1.4], the multiplicative monoid of R is locally finitely generated and thus taut.

If $\overline{R}$ is the ring of integers of an algebraic number field or a holomorphy ring in an algebraic function field for a finite set of places, then $\mathrm{Pic}(R)$ is finite, and every class (in particular the principal class) contains infinitely many primes (see [13, Theorem 2.10.14 and Corollary 2.11.16] for the number field case and [13, Theorem 8.9.5 and Proposition 8.9.7] for the function field case). □

Example 2.5. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. By [9, Corollary 3.10], $\mathcal{O}_K$ is fully elastic if its class number is p^k for some prime p and positive integer k. We extend this result to all algebraic rings of integers. If $\mathrm{Cl}(\mathcal{O}_K)$ denotes the ideal class group of $\mathcal{O}_K$, then [7, Lemma 3.2] implies that

$$\mathfrak{L}(\mathcal{O}_K) = \mathfrak{L}(\mathcal{B}(\mathrm{Cl}(\mathcal{O}_K))).$$

Since $\mathrm{Cl}(\mathcal{O}_K)$ is a finite abelian group, $\mathcal{B}(\mathrm{Cl}(\mathcal{O}_K))$ is finitely generated as a monoid, and hence by [1] has rational and accepted elasticity. Thus, so too does $\mathcal{O}_K$. By [16], each ideal class of $\mathrm{Cl}(\mathcal{O}_K)$ contains infinitely many prime ideals. Thus, $\mathcal{O}_K$ has infinitely many prime elements. Therefore Corollary 2.2 indicates that $\mathcal{O}_K$ is both fully elastic and asymptotically fully elastic.

Example 2.6. We extend slightly the result in Example 2.5. Let $\mathcal{O}$ be an order in the number field K with integral closure $\bar{\mathcal{O}}$ and suppose further that for every prime ideal P of $\mathcal{O}$ there is exactly one prime ideal of $\bar{\mathcal{O}}$ lying above P. By Theorem 2.4, $\rho(\mathcal{O})$ is rational and accepted and $\mathcal{O}$ contains a prime element. Corollary 2.2 again indicates that $\mathcal{O}$ is both fully elastic and asymptotically fully elastic.

Example 2.7. We take this example from [8]. Let V be a discrete valuation ring with quotient field K. We denote by p the unique irreducible element of V. Let L be a finite extension of K and B be the ring of integers of L over V. B is a principal ideal domain with finitely many irreducible elements. Suppose that p ramifies in B (i.e., $p = \pi^e$ for some irreducible $\pi \in B$ and integer $e \geq 2$). Let $R = V + xB[x]$. By [8, Theorem 3.3], $\rho(R) = \frac{e+1}{2}$ and is accepted. Moreover, [8, Theorem 3.1] guarantees the existence of a prime element in R. Hence R is both fully elastic and asymptotically fully elastic.

Example 2.8. [9, Corollary 3.10] indicates that a large class of *Krull monoids* (see [13, Chapter 2.5]) are fully elastic. It is not known in general if all Krull monoids are fully elastic. Yet here is an example of a relatively simple Krull monoid to which our Theorem applies. Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} < a_n$ be a sequence of integers with $n \geq 3$. Let

$$M = \{(x_1, \cdots, x_n, x_{n+1}) \mid a_1x_1 + \cdots + a_nx_n - a_nx_{n+1} = 0 \text{ with each } x_i \in \mathbb{N}_0\}.$$

By [10, Proposition 1.2], M is a Krull monoid with finitely many irreducible elements. Hence, by [1, Theorem 7] M is finitely generated and has accepted elasticity. Since the element $(0, \ldots, 0, 1, 1)$ is actually prime, Theorem 2.2 again indicates that M is both fully elastic and asymptotically fully elastic.

Example 2.9. Let $k_0 \subseteq k_1 \subseteq \cdots k_{n-1} \subsetneq K$ be a sequence of finite fields and set $R = k_0 + k_1 x + k_2 x^2 + \cdots + k_{n-1} x^{n-1} + x^n K[x]$. The ring R is an order in the factorial domain $K[x]$ with conductor $F = x^n K[x]$. The only prime ideal of R lying above the conductor is $xK[x] \cap R$ and the only prime ideal of $K[x]$ lying above this prime ideal is clearly $xK[x]$. By Theorem 2.4, $\rho(R)$ is rational and accepted. By the Dirichlet-Kornblum Theorem [17], there is a prime element of $K[x]$ of the form $f(x) = 1 + x^n + \sum_{i=n+1}^{t} a_i x^i$. Then $f(x) + F = 1 + F$ is a unit in R/F so by Lemma 2.3, $f(x)$ is a prime in R as well. By Theorem 2.2, R is both fully elastic and asymptotically fully elastic.

We may also obtain full elasticity and asymptotic full elasticity with other hypotheses, as the next theorem illustrates.

Theorem 2.10. *Let M be taut. If M contains a prime element, then M is fully elastic and asymptotically fully elastic.*

Proof. There is a sequence $\{f_1, f_2, f_3, \dots\}$ of elements in M such that $\lim_{i\to\infty} \rho(f_i) = \rho(M)$. Since M is taut, then there exists a sequence $n_1, n_2, \dots$ of nonnegative integers such that $\rho(f_i^{n_i}) = \overline{\rho}(f_i)$. But we also know that $\rho(f_i^{n_i}) \geq \rho(f_i)$, and therefore $\lim_{i\to\infty} \rho(f_i^{n_i}) = \rho(M)$ as well. Therefore by Lemma 2.1,

$$[1, \rho(M)) \cap \mathbb{Q} \supseteq \mathcal{R}(M) \supseteq \left(\bigcup_{i=0}^{\infty} [1, \rho(f_i^{n_i})] \right) \cap \mathbb{Q} = [1, \rho(M)) \cap \mathbb{Q}.$$

A similar argument works for $\overline{\mathcal{R}}(M)$. $\square$

However, even if M does not have accepted elasticity and is not taut, if it contains a prime then it may still be fully elastic. We demonstrate this with an example which will require the following lemma.

Lemma 2.11. *Let M be an atomic monoid containing a prime p, and let $f \in M$ be given. Then $\mathcal{R}(M) \supseteq \{\, a/b \in \mathbb{Q} \mid 1 \leq a/b \leq \rho(f),\ (a-b) | (L(f) - l(f)) \,\}$.*

Proof. Let a/b with $1 \leq a/b \leq \rho(f)$ and $(a-b)|(L(f) - l(f))$ be given. Let

$$\begin{aligned} j &= \frac{bL(f) - al(f)}{a-b} - l(f) \\ &= \frac{b(L(f) - l(f))}{a-b} + l(f). \end{aligned}$$

Clearly j is an integer under our assumptions; furthermore, since $1 \le a/b$, j is nonnegative. Now consider the element fp^j. We have

$$\begin{aligned} \rho(fp^j) &= \frac{L(f)+j}{l(f)+j} \\ &= \frac{aL(f)-bL(f)+bL(f)-al(f)}{al(f)-bl(f)+bL(f)-al(f)} \\ &= \frac{a(L(f)-l(f))}{b(L(f)-l(f))} \\ &= \frac{a}{b}. \end{aligned}$$

□

Example 2.12. Consider the multiplicative monoid M of $\mathbb{Z}+nx\mathbb{Z}[x]$. If $n = p_1 \cdots p_k$ is a squarefree integer (with each p_i prime), then $\rho(M) = k$ and M does not have accepted elasticity [14]. Again, by [14], we have that $\rho((nx)^m) = (k(m-1)+1)/m$. Therefore $\overline{\rho}(nx) = k$, but no m exists such that $\rho((nx)^m) = k = \overline{\rho}(nx)$. Thus M is not taut. But M is a submonoid of the unique factorization domain $\mathbb{Z}[x]$, which is taut.

By Lemma 2.3, $1+nx$ is prime in M. We now show that M is fully elastic. By [14], $\rho(M)$ has elasticity k. In addition, it is also shown in [14] that for $f = nx$, $L(f^i) = k(i-1)+1$ and $l(f^i) = i$. Therefore, $L(f^i) - l(f^i) = (k-1)(i-1)$. Note that since $\overline{\rho}(f) = k$, this signifies that, given a rational $1 \le a/b \le k$, there is an integer i such that $\rho(f^i) \ge a/b$ and $a-b \mid L(f^i) - l(f^i)$. That M is fully elastic now follows from Lemma 2.11.

3. Full Elasticity in Monoids Containing No Primes

We now consider monoids which do not contain any prime elements (see for instance Example 3.7) and hence our previous results do not apply. We open with an example involving direct calculation.

Example 3.1. Let $S \subseteq \mathbb{Z}$ be infinite and let

$$\mathrm{Int}(S,\mathbb{Z}) = \{\, f(x) \mid f(x) \in \mathbb{Q}[x] \text{ with } f(s) \in \mathbb{Z} \text{ for all } s \in S \,\}$$

represent the ring of polynomials in $\mathbb{Q}[x]$ which are integer-valued on S. By [2, Proposition 3.2], $\mathrm{Int}(S,\mathbb{Z})$ contains no prime elements and by [11, Theorem 4.5], $\mathrm{Int}(S,\mathbb{Z})$ is fully elastic. We argue that $\mathrm{Int}(S,\mathbb{Z})$ is also asymptotically fully elastic.

By [11, Proposition 3.4], for every prime number p, there exists a sequence $i_1, i_2, \ldots, i_t$ of integers such that the polynomial

$$f_p(x) = \frac{(x-i_1)(x-i_2)\cdots(x-i_t)}{p}$$

is irreducible in $\mathrm{Int}(S,\mathbb{Z})$. Set

$$h_p(x) = (x-i_1)\cdots(x-i_t).$$

It is easy to see by the above that $L(h_p(x)^k) = kt$ and $l(h_p(x)^k) = 2k$ for any $k \geq 1$. By [11, Lemma 4.3], for any nonnegative integers k and s we have

$$\mathcal{L}\left(h_p^k(x)f_p^s(x)\right) = \{2j + (k-j)t + s \,|\, 0 \leq j \leq k\} \tag{3.1}$$

and hence $\rho\left(h_p^k(x)f_p^s(x)\right) = \frac{kt+s}{2k+s}$. Notice that (3.1) implies for each positive integer n that

$$\mathcal{L}\left(((h_p^k(x)f_p^s(x))^n\right) = \{2j + (nk-j)t + ns \,|\, 0 \leq j \leq nk\}$$

and thus $\rho\left(((h_p^k(x)f_p^s(x))^n\right) = \frac{n(kt+s)}{n(2k+s)} = \rho\left(h_p^k(x)f_p^s(x)\right)$. Hence $\overline{\rho}\left(h_p^k(x)f_p^s(x)\right) = \rho\left(h_p^k(x)f_p^s(x)\right)$ for all $k, s \geq 0$. Since these are the polynomials used to show full elasticity in [11], we have full asymptotic elasticity as well.

In general, the calculation of the previous example is not always practical. Hence, we isolate elements which will act as tools in elasticity arguments.

Definition 3.2. Call an element $f \in M$ *flexible* if for all $j \in \mathbb{N}$ there exists a sequence $p_1, p_2, \ldots, p_j$ of irreducibles in M such that $L(fp_1p_2\cdots p_j) = L(f) + j$ and $l(fp_1p_2\cdots p_j) = l(f) + j$.

If a prime p exists in M, then every $f \in M$ is flexible—simply take each $p_i = p$. It follows from Example 3.1 that each $h_p(x)$ is flexible. However, even when no primes exist, flexibility allows one to create arbitrarily long finite sequences of elements in M such every element in the sequence acts "independently" of the elements that appear before it. In contrast to Example 3.1, in the next example we produce a monoid with no irreducible flexible elements.

Example 3.3. Consider an additive primitive numerical monoid

$$S = \{x_1n_1 + \cdots + x_kn_k \mid x_i \in \mathbb{Z} \text{ with } x_i \geq 0\} =: \langle n_1, \ldots, n_k\rangle$$

minimally generated by $n_1, \ldots, n_k$ with $k \geq 2$. We begin by showing that all large elements in S have factorizations of varying lengths. To see this, let $N = \sum_{i=1}^k n_in_k$ and suppose that $n \in S$ with $n > N$. Write $n = \sum_{i=1}^k y_in_i$. Clearly some $y_t > n_k$ (otherwise $n \leq N$). We claim that $\rho(n) > 1$. If $t < k$ then $n = \left(\sum_{i\neq t,k} y_in_i\right) + (y_t - n_k)n_t + (y_k + n_t)n_k$ and $\rho(n) \geq (\sum_{i=1}^k y_i)/((\sum_{i=1}^k y_i) + (n_t - n_k))) > 1$. If $t = k$ then write $n = \left(\sum_{i\neq 1,k} y_in_i\right) + (y_1 + n_k)n_1 + (y_k - n_1)n_k$ and again $\rho(n) > 1$.

Let $f = n_i$, a generator of S. We show that f is not flexible. First, notice that any irreducible $p \in S$ must satisfy $p \geq n_1$. Assume f is flexible and let $j > N/n_1$. Let $p_1, p_2, \ldots, p_j$ be the sequence of irreducibles in M such that $L(f + p_1 + p_2 + \cdots + p_j) = L(f) + j = j + 1$ and $l(f + p_1 + p_2 + \cdots + p_j) = l(f) + j = j + 1$ (and hence the elasticity is 1). But this is a contradiction, since $f + p_1 + p_2 + \cdots + p_j > N$, so its elasticity must be strictly greater than 1. Therefore f is not flexible.

Theorem 3.4. *Let M be an atomic monoid containing an element f such that $\rho(f) = \overline{\rho}(f)$. If f^i is flexible for all positive integers i, then $\mathcal{R}(M) \supseteq [1, \rho(f)] \cap \mathbb{Q}$.*

Proof. Similar to the proof of Lemma 2.1, let $a/b \in [1, \rho(f)] \cap \mathbb{Q}$ be given. Let $i = a - b$ and $j = L(f)b - l(f)a$. These quantities are both nonnegative since $1 \leq a/b \leq \rho(f)$. Since f^i is flexible, there exists a sequence $p_1, p_2, \dots, p_j$ in M such that $L(f^i p_1 p_2 \cdots p_j) = L(f^i) + j = iL(f) + j$ and $l(f^i p_1 p_2 \cdots p_j) = l(f^i) + j = il(f) + j$. By our choice of i and j, $\rho(f^i p_1 p_2 \cdots p_j) = a/b$. Therefore $\mathcal{R}(M) \supseteq [1, \rho(f)] \cap \mathbb{Q}$. □

Corollary 3.5. *Let M be an atomic monoid with accepted elasticity and choose $f \in M$ such that $\rho(f) = \rho(M)$. If f^i is flexible for all positive integers i, then M is fully elastic.*

Theorem 3.6. *Let M be taut. Let $s = \{f_1, f_2, f_3 \dots\}$ be a sequence of elements of M such that* $\lim_{i\to\infty} \rho(f_i) = \rho(M)$ *and $\{n_1, n_2, n_3, \dots\}$ a sequence of positive integers such that $\rho(f_i^{n_i}) = \overline{\rho}(f_i)$ for all i. If $g_i = f_i^{n_i}$ and each g_i^j is flexible for all positive integers j, then M is fully elastic.*

Proof. If M has accepted elasticity then we may just use Corollary 3.5. So let us suppose the contrary. Clearly the sequence $\{\rho(g_1), \rho(g_2), \rho(g_3), \dots\}$ also converges to $\rho(M)$ and $\rho(g_i) \neq \rho(M)$. Therefore by Theorem 3.4,

$$[1, \rho(M)) \cap \mathbb{Q} \supseteq \mathcal{R}(M) \supseteq \left(\bigcup_{i=1}^{\infty} [1, \rho(g_i)] \right) \cap \mathbb{Q} = [1, \rho(M)) \cap \mathbb{Q}.$$

□

We illustrate the previous three results with an example.

Example 3.7. Consider the multiplicative monoid M of $K[x^2, x^3]$, where K is an algebraically closed field of characteristic $p > 0$. Every $f(x) \in M$ factors uniquely in $K[x]$ as a product $ux^n \prod_{i=1}^k (1 + a_i x)$ of linear polynomials and a unit u. Note that such a product falls in M precisely when $n \geq 2$ or when $n = 0$ and $\sum_{i=1}^k a_i = 0$.

We first show M has no primes. Suppose, to the contrary, that $f(x)$ is prime in M. Then $f(x)$ divides $f(x)f(-x) = u^2[x^2]^n \prod_{i=1}^k [1 - a_i^2 x^2]$, so by primality, $f(x)$ divides x^2 or $1 - a_i^2 x^2$ for some i. But $f(x)$ has order at least 2, so $f = x^2$ or $f = 1 - a_i^2 x^2$. Clearly, $x^6 = x^3 \cdot x^3$ shows that x^2 is not prime in M, so $f = 1 - a^2 x^2$ for some nonzero $a \in K$. But $1 - a^2 x^2$ is not prime by Lemma 2.3, and hence M contains no primes.

We show that M is taut. Let $f = x^n \prod_{i=1}^k (1 + a_i x) \in M$ be given. The a_i finitely generate a subfield F of K; for each $t \geq 1$, every factor of f^t will be a polynomial over F. Let g be an irreducible factor of f in M, where

$$g = (x^n)^e \prod_{i=1}^k (1 + a_i x)^{e_i}$$

for some $e, e_i \geq 0$. Note that $ne = 0, 2,$ or 3, for otherwise g factors in M as $x^2(g/x^2)$. If $ne = 2$ or 3, then no subproduct g' of $\prod_{i=1}^k (1 + a_i x)^{e_i}$ can be in M

for otherwise g factors in M as $g'(g/g')$. By the first paragraph of this example, this requires that $S = \{a_1, \dots, a_k\}$ is a zero-free sequence in $(F, +)$, where each a_i appears e_i many times in S. Since F has characteristic p, the a_i generate a finite subgroup of $(F, +)$. Thus the possible S are finite in number and there are only finitely many irreducible g with $ne \neq 0$ that divide some f^t. Now suppose $ne = 0$. Then $\sum_{i=1}^k e_i a_i = 0$, so $S = \{a_1, \dots, a_k\}$ is a zero sequence in $(F, +)$, where each a_i appears e_i many times in S. If S is not a minimal zero sequence, then it consists of two zero subsequences S_1 and S_2. For $j = 1, 2$, set $g_j = \prod_{i=1}^k (1 + a_k x)^{h_{i,j}}$, where $h_{i,j}$ is the number of times a_i appears in S_j. Since each S_j is a zero sequence, each $g_j \in M$ and so g is not irreducible in M. Therefore, when $ne = 0$, g is irreducible only if S is a minimal zero sequence in $(F, +)$. Since the a_i generate a finite subgroup of $(F, +)$, there are only finitely many minimal zero sequences of the form S. Thus we have shown $\mathcal{U}(f)$ is finite for each $f \in M$, so M is taut by the commentary in section 1.

Next, we show that $\mathcal{R}' = \{\rho(f) \mid f \in M,\ f(0) \neq 0\}$ is unbounded. Note that if $q \neq p$ is prime, then $ax^q + 1$ is irreducible in M for every nonzero $a \in K$. To see this, let α be a root of $ax^q + 1$ in K; then all the roots of $ax^q + 1$ can be enumerated as $\alpha, \zeta_q \alpha, \dots, \zeta_q^{q-1}\alpha$, where ζ_q is some qth root of unity. Since $q \neq p$ is prime, these roots are all distinct, and $ax^q + 1$ factors in $K[x]$ as $\prod_{i=0}^{q-1}(x - \zeta_q^i \alpha)$. Any proper nonempty subproduct will have a linear coefficient of the form $\alpha(\zeta_q^{i_1} + \ldots + \zeta_q^{i_k})$ for some $0 \leq i_1 < \ldots < i_k \leq q-1$ and $1 \leq k \leq q-1$. This linear coefficient cannot be 0 since any proper subset of $\{1, \zeta_q, \dots, \zeta_q^{q-1}\}$ is linearly independent over $\mathbb{F}_p$ and so $ax^q + 1$ must be irreducible. Let $\{q_1, q_2, \ldots\}$ be a strictly increasing sequence of prime numbers such that each $q_i > p$ and $q_i \in S$. Then in M we have the following two factorizations into irreducibles:

$$\begin{aligned} x^{q_1 q_k} - 1 &= (x^{q_1} - 1)(\zeta_{q_k} x^{q_1} - 1) \cdots (\zeta_{q_k}^{q_k - 1} x^{q_1} - 1) \\ &= (x^{q_k} - 1)(\zeta_{q_1} x^{q_k} - 1) \cdots (\zeta_{q_1}^{q_1 - 1} x^{q_k} - 1). \end{aligned}$$

so $\rho(x^{q_1 q_k} - 1) \geq q_k/q_1$. Hence $\sup \mathcal{R}' = \infty$.

Lastly, we show that any $f \in M$ with $f(0) \neq 0$ is flexible. Let an arbitrary such f be given (without loss, $f(0) = 1$) and factor it in $K[x]$ as $\prod_{i=1}^k (1 + a_i x)$. Let F be the subfield of K generated by the a_i, and choose $\alpha \in K$ such that $[F(\alpha) : F] > p$. Set $g = (1 + \alpha x)^p$, which is irreducible in M. We claim $L(fg) = L(f) + 1$ and $l(fg) = l(f) + 1$. Suppose h is an irreducible factor of fg in M; then $h = f'g'$ where f' divides f in $K[x]$ and g' divides g in $K[x]$. The linear term of h is 0, which is the sum of the linear terms of f' and g'. If $g' \neq 1$, then its linear term is $r\alpha$ for some $0 < r < p$. If f' also has a nonzero linear term, then this implies that α satisfies a non-trivial linear equation over F, a contradiction to the choice of α. Thus $f' \in M$; so h now factors in M as $f'(h/f')$. By irreducibility of h, $f' = 1$. Thus $h = g'$, which necessarily implies $h = g$. Hence every factorization of fg contains g as one of its irreducible factors and $L(fg) = L(f) + 1$ and $l(fg) = l(f) + 1$. Since $f \in M$ with $f(0) \neq 0$ was arbitrary, this shows that all such f are flexible.

Thus we have shown all the necessary conditions to apply Theorem 3.6; every $K[x^2, x^3]$ is fully elastic for K an algebraically closed field of characteristic $p > 0$.

4. Elastic Properties of $K[[x; S]]$

We close by extending a result of [9] where the authors offer the power series subring $K[[x^n, x^{n+1}, \ldots, x^{2n-1}]]$ (with K any field) as an example of an integral domain with finite elasticity which is not fully elastic. We show that this holds for any such ring of the form $M = K[[x^{n_1}, x^{n_2}, \ldots, x^{n_k}]] =: K[[x; S]]$ where $S = \langle n_1, \ldots, n_k \rangle$ is a primitive numerical monoid with Frobenius number $\mathcal{F}(S)$. We will require a lemma taken from [9].

Lemma 4.1 (Theorem 2.2 of [9]**).** *Let $S = \langle n_1, \ldots, n_t \rangle$ be a numerical monoid, where $n_1, \ldots, n_t \in \mathbb{N}$ minimally generate S and $t \geq 2$. Then S is not fully elastic. In particular, there exists $N \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$ where $\alpha > 1$, such that for all $n \in S$ where $n > N$, $\rho(n) > \alpha$.*

Example 4.2. Using Lemma 4.1, we extend Example 3.3 and show that no element in a primitive numerical monoid is flexible. Let S, N and α be as in Lemma 4.1 and let $f \neq 0$ be an arbitrary element of S which we assume is flexible. For f, one can always pick a j large enough such that $f + p_1 + p_2 + \cdots + p_j > N$ where $p_1, p_2, \ldots, p_j$ are irreducibles in S with $L(f + p_1 + p_2 + \cdots + p_j) = L(f) + j$ and $l(f + p_1 + p_2 + \cdots + p_j) = l(f) + j$. Moreover, once can pick large j such that $(L(f)+j)/(l(f)+j) < \alpha$. Hence, $\rho(f+p_1+p_2+\cdots+p_j) = (L(f)+j)/(l(f)+j) < \alpha$, a contradiction.

If $f(x) = \sum_{s \in S} f_s x^s \in K[[x; S]]$, then set $\operatorname{ord}(f(x)) = \min\{s \mid s \geq 0 \text{ and } f_s \neq 0\}$. We can observe the following: 1) a nonzero element $f(x)$ with $\operatorname{ord}(f(x)) = 0$ is a unit in M, 2) any irreducible element $f(x) \in M$ must satisfy $n_1 \leq \operatorname{ord}(f(x)) \leq \mathcal{F}(S) + n_1$ and $\operatorname{ord}(f(x)) \in S$, 3) Any element $f(x)$ with $\operatorname{ord}(f(x)) = n_i$ is irreducible in M. We begin with a lemma.

Lemma 4.3. *Let $n \in \mathbb{N}$ be given. Let $T = \{\, f(x) \in M \mid 0 < \operatorname{ord}(f(x)) < n \,\}$. Then $\mathcal{R}(T)$ is a finite set.*

Proof. Let $m = \operatorname{ord}(g)$ for some $g \in T$. Consider a factorization of g in M, say $g(x) = uf_1 \cdots f_t$, where u is a unit in M and each f_i is irreducible in M. Clearly then $\operatorname{ord}(f_i) \in S$, $n_1 \leq \operatorname{ord}(f_i) \leq \mathcal{F}(S) + n_1$ and $m = \sum_{i=1}^{t} \operatorname{ord}(f_i)$. Thus t, the length of this factorization, must be a positive integer satisfying the inequalities: $tn_1 \leq m \leq t(\mathcal{F}(S) + n_1)$. Thus for any $g \in T$ with $\operatorname{ord}(g) = m$, we have

$$\mathcal{L}(g) \subseteq \left\{ t \in \mathbb{N} \,\middle|\, n_1 \leq \frac{m}{t} \leq \mathcal{F}(S) + n_1 \right\},$$

which is a finite set. Thus there are only finitely many possibilities for $\rho(g)$ for g of order m, and varying over the finitely many values of m we obtain the desired result. □

Theorem 4.4. *Let S be a nontrivial numerical monoid and K a field. Then $K[[x; S]]$ is* not *fully elastic.*

Proof. Let N and α be as defined in Lemma 4.1. Fix $n > \max(N+\mathcal{F}(S)+1, (2n_t+1)\mathcal{F}(S)+1)$ and consider $U = \{ f \in M \mid \mathrm{ord}(f) \geq n \}$. Let $g \in U$ and $m = \mathrm{ord}(g)$.

In $K[[x]]$, we see that $g = ux^m$ where u has order 0. If $u \in K[[x; S]]$, then g is an associate of x^m in $K[[x; S]]$. Thus $\mathcal{L}_M(g) = \mathcal{L}_S(m)$ and $\rho_M(g) = \rho_S(m) > \alpha$ by the assumption on $m \geq n$.

If $u \notin K[[x; S]]$, then let k be the least positive integer such that $x^k u \in K[[x; S]]$. Clearly $k \leq \mathcal{F}(S) + 1$. Factor $x^k u$ into irreducibles in $K[[x; S]]$; then $x^k u = \prod_{i=1}^r x^{k_i} v_i$. Each k_i is in S with $n_1 \leq k_i \leq k$ and each v_i is not a unit of $K[[x; S]]$ by the minimality of k. Note that $r \leq k/n_1$ and that since $m > (2n_t + 1)\mathcal{F}(S) + 1$, we have $m - k > \mathcal{F}(S) + 1$ and $m - k \in S$.

Thus g factors as $(x^{m-k}) \prod_{i=1}^r x^{k_i} v_i$ and so:

$$L_M(g) \geq L_S(m-k) + r$$

$$l_M(g) \leq l_S(m-k) + r.$$

As a result, $\rho_M(g) \geq (L_S(m-k) + r)/(l_S(m-k) + r)$. But

$$\begin{aligned}
\rho_S(m-k) - \frac{L_S(m-k)+r}{l_S(m-k)+r} &= \frac{rL_S(m-k) - rl_S(m-k)}{l_S(m-k)^2 + rl_S(m-k)} \\
&< \frac{rL_S(m-k) - rl_S(m-k)}{l_S(m-k)^2} = \frac{r\rho_S(m-k) - r}{l_S(m-k)} \\
&\leq \frac{\rho_S(m-k) - 1}{2} = \frac{\rho_S(m-k)}{2} - \frac{1}{2}
\end{aligned}$$

The last inequality follows since $l_S(m-k)/r \geq 2$. Indeed, $r \leq k/n_1 \leq \mathcal{F}(S)$ and $m - k \geq 2n_t\mathcal{F}(S)$ so $l_S(m-k) \geq 2\mathcal{F}(S)$. From the above inequalities we immediately see that $\rho_M(g) > \rho_S(m-k)/2 + 1/2 > \alpha/2 + 1/2 > 1$ by the assumption that $m - k > N$. Therefore in both cases, we have concluded that $\mathcal{R}(U) \cap [1, \alpha/2 + 1/2] = \emptyset$. In addition, by Lemma 4.3, $\mathcal{R}(T)$ is a finite set, where T is the set of elements of order less than n. Since $[1, \alpha/2+1/2] \cap \mathbb{Q}$ is infinite and $\mathcal{R}(K[[x; S]]) = \mathcal{R}(T) \cup \mathcal{R}(U)$, $K[[x; S]]$ cannot be fully elastic. □

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