ASYMPTOTIC ELASTICITY IN ATOMIC MONOIDS

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ABSTRACT. Let M be a commutative atomic monoid (i.e. every nonzero nonunit of M can be factored as a product of irreducible elements). Let $\rho(x)$ denote the elasticity of $x \in M$, $\mathcal{R}(M) = \{\rho(x) \mid x \in M\}$ the set of elasticities of elements in M, and $\rho(M) = \sup \mathcal{R}(M)$ the elasticity of M. Define $\bar{\rho}(x) = \lim_{n\to\infty} \rho(x^n)$ to be the *asymptotic elasticity* of x. We determine some basic properties of the function $\bar{\rho}$ and determine its image when M is a block monoid.

Let M be a commutative cancellative monoid with M^* its set of nonunits and $\mathcal{A}(M)$ its set of irreducibles (or atoms). We suppose M is *atomic* (i.e. every element of M^* is a sum of atoms). Much recent literature has been devoted to the study of atomic monoids in which elements fail to factor uniquely. In particular, a central topic has been the *elasticity* of elements of M, which measures the possible deviation in lengths of factorizations into irreducibles. We study here an asymptotic variation of the elasticity. We begin with some definitions and notations.

For $x \in M^*$, define $\mathcal{L}(x) = \{n \mid x = \alpha_1, \ldots, \alpha_n \text{ with each } \alpha_i \in \mathcal{A}(M)\}$ to be the set of lengths of factorizations of x into irreducibles and $\mathcal{L}(M) = \{\mathcal{L}(x) \mid x \in M^*\}$ to be the set of lengths of M. Define $L(x) = \sup \mathcal{L}(x)$ and $l(x) = \inf \mathcal{L}(x)$, and further define $\rho(x) = \frac{L(x)}{l(x)}$ to be their quotient. $\rho(x)$ is called the *elasticity* of x. If for each $x \in M^*$ the set $\mathcal{L}(x)$ is finite, then M is called a *bounded factorization monoid* (BFM) and in this case each $\rho(x)$ is rational. We also define $\mathcal{R}(M) = \{\rho(x) \mid x \in M^*\}$ to be the set of elasticities of nonunits in M, and $\rho(M) = \sup \mathcal{R}(M)$ to be the *elasticity* of M. If $\rho(M) = 1$, then M is said to be *half-factorial*. Basic facts and background on the study of elasticity can be found in [2] and [13]. If M is finitely generated, then [1, Theorem 7] implies that $\rho(M) = \frac{m}{n} \in \mathbb{Q}$ and $\alpha_1 \ldots \alpha_m = \beta_1 \ldots \beta_n$ for some irreducibles $\alpha_i, \beta_j \in \mathcal{A}(M)$.

The work completed here is motivated by an earlier paper [7] written by three of the current four authors where the set $\mathcal{R}(M)$ is studied when M is a BFM. In particular, an atomic monoid M is called *fully elastic* (or fe) if $\mathcal{R}(M) = [1, \rho(M)] \cap \mathbb{Q}$ (in the case where $\rho(M) = \infty$, then $\mathcal{R}(M) = [1, \infty) \cap \mathbb{Q}$). Our results in [7] indicate that a large class of Krull monoids are fully elastic, while numerical monoids which require more than one generator are not. Full elasticity has been explored recently in other works (see [8] and [4]).

For all $x \in M^*$, define $\bar{L}(x) = \lim_{n\to\infty} \frac{L(x^n)}{n}$, and $\bar{l}(x) = \lim_{n\to\infty} \frac{l(x^n)}{n}$. From [3], we know both these limits exist, although $\bar{L}(x)$ may be infinite. For $x \in M^*$, we define $\bar{\rho}(x) = \frac{\bar{L}(x)}{\bar{l}(x)} = \lim_{n\to\infty} \rho(x^n)$ to be the *asymptotic elasticity* of x, which exists since both $\bar{L}(x)$ and $\bar{l}(x)$ do. Under the assumption that M is finitely generated, it is shown in [1, Theorem 12] and [10, Theorem 2] that $\bar{L}(x)$ and $\bar{l}(x)$ are nonzero and rational for all elements $x \in M^*$, and hence $\bar{\rho}(x)$ is rational as well. Define $\overline{\mathcal{R}}(M) = \{\bar{\rho}(x) \mid x \in M^*\}$ to be the set of asymptotic elasticities, and $\bar{\rho}(M) = \sup \overline{\mathcal{R}}(M)$ to be the asymptotic elasticity of M. We begin with several basic properties of $\bar{\rho}$.

Lemma 1. *If $x \in M^*$, then $\bar{\rho}(x) \geq \rho(x^n) \geq \rho(x)$ for all $n \in \mathbb{N}$. Moreover, $\bar{\rho}(M) = \rho(M)$.*

Proof. It is easy to verify that $L(x_1x_2) \geq L(x_1) + L(x_2)$ and $l(x_1x_2) \leq l(x_1) + l(x_2)$ for all $x_1, x_2 \in M^*$. It follows that $L(x^k) \geq kL(x)$ and $l(x^k) \leq kl(x)$ for all $k \in \mathbb{N}$, from which $\rho(x^k) = \frac{L(x^k)}{l(x^k)} \geq \frac{kL(x)}{kl(x)} = \rho(x)$. Thus for all $x \in M^*$, $\bar{\rho}(x) = \lim_{k\to\infty} \rho(x^k) \geq \rho(x)$. For all $n \in \mathbb{N}$, $x^n \in M^*$ so

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$\bar{\rho}(x) = \lim_{k\to\infty} \rho(x^k) = \lim_{k\to\infty} \rho(x^{kn}) = \bar{\rho}(x^n) \geq \rho(x^n) \geq \rho(x)$, which completes the proof of the first statement. For the second statement, every $r \in \overline{\mathcal{R}}(M)$ is the limit of some sequence in $\mathcal{R}(M)$. Hence $\bar{\rho}(M) \leq \rho(M)$. It follows from (i) that $\bar{\rho}(M) = \sup\{\bar{\rho}(x) \mid x \in M^*\} \geq \sup\{\rho(x) \mid x \in M^*\} = \rho(M)$, completing the proof. □

In the next theorem, we characterize elements $x \in M^*$ for which $\rho(x) = \bar{\rho}(x)$.

Theorem 2. *If $x \in M^*$, then the following statements are equivalent:*

(i) $\bar{\rho}(x) = \rho(x)$,
(ii) $\rho(x^n) = \rho(x)$ *for all* $n \in \mathbb{N}$,
(iii) *there is an integer* $m \geq 2$ *such that* $\rho(x^{m^n}) = \rho(x)$ *for infinitely many* $n \in \mathbb{N}$,
(iv) *for all* $n \in \mathbb{N}$, $L(x^n) = nL(x)$ *and* $l(x^n) = nl(x)$.

Proof. From Lemma 1, $\bar{\rho}(x) \geq \rho(x^n) \geq \rho(x)$ for all $n \in \mathbb{N}$. It follows that if $\bar{\rho}(x) = \rho(x)$ then $\rho(x^n) = \rho(x)$ for all $n \in \mathbb{N}$, so (i) implies (ii). That (ii) implies (iii) is obvious. Finally, if (iii) holds then $\bar{\rho}(x) = \lim_{k\to\infty} \rho(x^k) = \lim_{k\to\infty} \rho(x^{m^k}) = \rho(x)$, so (iii) implies (i). Clearly (iv) implies (ii). If (ii) hold, then suppose that either $L(x^n) > nL(x)$ or $l(x^n) < nl(x)$. Then $\rho(x^n) = \frac{L(x^n)}{l(x^n)} > \frac{nL(x)}{nl(x)} = \rho(x)$, a contradiction. Thus (iv) holds and the proof is complete. □

In light of the last two results, we make the following definition.

Definition 3. Let M be an atomic monoid. If $\bar{\rho}(M) < \infty$, then call M *asymptotically fully elastic* (or *afe*) if $\overline{\mathcal{R}}(M) \supseteq [1, \bar{\rho}(M)] \cap \mathbb{Q}$. Otherwise, M is afe if $\overline{\mathcal{R}}(M) \supseteq [1, \infty) \cap \mathbb{Q}$. An atomic integral domain D is afe if its multiplicative monoid (denoted $D^\bullet$) is afe.

By placing some assumptions on the monoid M, we can say more about the afe property. The following definition is suggested by several of the results in [12].

Definition 4. A BFM M is called a $\mathcal{U}$-monoid if for all $x \in M^*$ there exists an integer $\alpha \in \mathbb{N}$ (depending on x) such that $\bar{\rho}(x) = \rho(x^\alpha)$.

For each $x \in M^*$ set $\mathcal{U}(x) = \{y \mid y \in \mathcal{A}(M) \text{ and } y|x^n \text{ for some } n \in \mathbb{N}\}$.

Lemma 5 (Theorem 3.8.1, [12])**.** *Let M be a BFM. If for each $x \in M^*$ the set $\mathcal{U}(x)$ contains finitely many elements (up to associates), then M is a $\mathcal{U}$-monoid.*

Several familar classes of monoids are $\mathcal{U}$-monoids, including (1) finitely generated monoids, (2) monoids with finitely many irreducible elements which are not prime, and (3) the class of Krull monoids. Notice that under the hypothesis M is a $\mathcal{U}$-monoid, we are assured that the elements of $\overline{\mathcal{R}}(M)$ are all rationals. Lemma 5 immediately implies the following corollary.

Corollary 6. *Let M be a $\mathcal{U}$-monoid. Then $\overline{\mathcal{R}}(M) \subseteq \mathcal{R}(M)$ and hence, if M is afe then M is fe.*

Example 7. By the corollary, a finitely generated monoid which is not fe cannot be afe. This is the case for numerical monoids requiring more than one generator [7, Theorem 2.2]. It is of interest to note that the containment in Corollary 6 can be proper. Let S be a numerical monoid which is not cyclic. Let $a_1, \ldots, a_t \in \mathbb{N}$ be a minimal set of generators for the numerical monoid $S = \langle a_1, \ldots, a_t \rangle$, where $t \geq 2$. Suppose $1 < a_1 < \ldots < a_t$ and let $n \in S$. Then $n^{a_1 a_t} = (a_1 n)a_t = (a_t n)a_1$ are the minimal and maximal length factorizations of $n^{a_1 a_t}$, respectively, so $\rho(n^{a_1 a_t}) = \frac{a_t n}{a_1 n} = \frac{a_t}{a_1}$. By Lemma 1, $\frac{a_t}{a_1} = \rho(n^{a_1 a_t}) \leq \bar{\rho}(n^{a_1 a_t}) = \bar{\rho}(n) \leq \bar{\rho}(S) = \rho(S) = \frac{a_t}{a_1}$ so $\bar{\rho}(n) = \frac{a_t}{a_1}$ and hence $\overline{\mathcal{R}}(S) = \{\frac{a_t}{a_1}\}$. From the proof of [7, Corollary 2.3], there are elasticities in $\mathcal{R}(S)$ less than $\frac{a_t}{a_1}$, so in particular the previous argument tells us that $\overline{\mathcal{R}}(S) \subsetneq \mathcal{R}(S)$.

Numerical monoids can be used to construct examples of integral domains which are not afe. Let $R = K[[X^n, \ldots, X^{2n-1}]]$ where $n \geq 2$ and K is any field. For $f \in R$, let $\text{ord}(f)$ represent the smallest power of X in f with a nonzero coefficient. The map $\varphi : R\backslash\{0\} \to S = \langle n, n+1, \ldots, 2n-1 \rangle$ defined by $\varphi(f) = \text{ord}(f)$ is a transfer homomorphism (see [11]) and hence R and S agree on the invariants ρ, $\mathcal{R}$ and $\overline{\mathcal{R}}$.

We focus in the remainder of this work on the afe property in block monoids. Let G be a nontrivial finitely generated abelian group written canonically in the form $G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$. If G is finite (i.e. $r = 0$) then we call k the *rank* of G. Let $\mathcal{F}(G)$ represent the free abelian monoid on G, and write the elements of $\mathcal{F}(G)$ as $\prod_{g\in G} g^{n_g}$. The *length* of $F = \prod_{g\in G} g^{n_g} \in \mathcal{F}(G)$ is defined to be $\sum_{g\in G} n_g$. The cross number of $F \in \mathcal{F}(G)$ is $\Bbbk(F) = \sum_{g\in G} \frac{n_g}{|g|}$ where $\mid g \mid$ represents the order of g in G. Set $\mathcal{B}(G) = \{\prod_{g\in G} g^{n_g} \mid \prod_{g\in G} g^{n_g} \in \mathcal{F}(G) \text{ and } \sum_{g\in G} n_g g = 0\}$. $\mathcal{B}(G)$ is known as the block monoid over G and its elements are sometimes referred to as *zero-sum sequences*. If G is finite, then is easy to argue that $\mathcal{B}(G)$ is finitely generated. In this case, the irreducible elements of $\mathcal{B}(G)$ are known as *minimal zero-sum sequences*. The *Davenport constant* of G, denoted by $D(G)$, is defined to be the maximum length of an irreducible in $\mathcal{B}(G)$. It is easy to argue that $D(G) \le |G|$ and if $G \cong \mathbb{Z}_n$ then $D(G) = n$. For $G \cong \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$ we define $M(G) = 1 + \sum_{i=1}^{k}(n_i - 1)$. In general, we have $D(G) \ge M(G)$, and $D(G) = M(G)$ if G is a p-group, a group of rank less than 3, or $|G| < 96$. It is also known that $\rho(\mathcal{B}(G)) = \frac{D(G)}{2}$. For a survey of known results concerning the Davenport constant, see [5].

We now present a result that relates asymptotic elasticity in $\mathcal{B}(G)$ to the cross number $\Bbbk$.

Theorem 8. *Let G be a finite abelian group. Let $x \in \mathcal{B}(G)$ and $y \in \mathcal{A}(\mathcal{B}(G))$. Then*

(i) $\bar{\rho}(y) \ge \max\{\Bbbk(y), \frac{1}{\Bbbk(y)}\}$,
(ii) $\bar{\rho}(x) = 1$ *if and only if every irreducible divisor α of the collective powers of x has $\Bbbk(\alpha) = 1$.*

Proof. (i) Let $n \in \mathbb{N}$. Since y^n is a product of n irreducibles, we know $l(y^n) \le n \le L(y^n)$. Dividing through by n and taking limits yields $\bar{l}(y) \le 1 \le \bar{L}(y)$. From [9, Proposition 7], we also know $\bar{l}(y) \le \Bbbk(y) \le \bar{L}(y)$. Taking the appropriate quotients yields the desired result. (ii) For $x \in \mathcal{B}(G)^*$ let S_x denote the monoid of all $u \in \mathcal{B}(G)$ dividing some power x^n. It follows easily that $\bar{\rho}(x) = 1$ if and only if S_x is half-factorial. The proof can now be completed using the cross number criteria in [6, Proposition 5.4]. □

As with the fe property in [7], we now show a large class of block monoids are afe for G finite.

Theorem 9. *Let G be a finite abelian group. Then $\overline{\mathcal{R}}(\mathcal{B}(G)) \supseteq \mathbb{Q} \cap [1, \frac{M(G)}{2}]$. Hence, if $M(G) = D(G)$, then $\overline{\mathcal{R}}(\mathcal{B}(G)) = \mathbb{Q} \cap [1, \frac{D(G)}{2}]$ and $\mathcal{B}(G)$ is both fe and afe.*

Proof. If $M(G) = D(G)$, then $\mathcal{B}(G)$ is fe by [7, Theorem 3.9]. We break the current proof into two cases. First, suppose that $G = \mathbb{Z}_n$ is cyclic (hence $M(G) = n$). If $B = (1^n)^u((n-1)^n)^v \in \mathcal{B}(\mathbb{Z}_n)$ for u and $v \in N$, then the proof of [7, Lemma 3.6] indicates that $\rho(B^k) = \frac{(n-1)ku+kv}{ku+kv} = \frac{(n-1)u+v}{u+v} = \rho(B)$ for any $k \in \mathbb{N}$. By Theorem 2, $\bar{\rho}(B) = \rho(B)$. If $x = \frac{p}{q}$ is any rational with $1 \le x \le \frac{n}{2}$, then setting $u = p - q$ and $v = (n-1)q - p$ yields $\rho(B) = x$ (that $u \le v$ follows from $\frac{p}{q} \le \frac{n}{2}$). The result for the cyclic case now follows.

Suppose G is not cyclic. Let $\alpha = g_1^{x_1} \ldots g_t^{x_t}$ be an irreducible in $\mathcal{B}(G)$ of length $M(G)$, where the g_i are all distinct. By the definition of $M(G)$, this can be done so that $x_1 = \mid g_1 \mid -1$. Viewing $\mathbb{Z}_{|g_1|}$ as a subgroup of G, our argument above can be used to show that $\overline{\mathcal{R}}(\mathcal{B}(G)) \supseteq \overline{\mathcal{R}}(\mathcal{B}(\mathbb{Z}_{|g_1|}) \supseteq \mathbb{Q} \cap [1, \frac{|g_1|}{2}]$. To complete the proof for this case, let $\bar{\alpha} = (-g_1)^{x_1} \ldots (-g_t)^{x_t}$, $\beta = g_1^{|g_1|}$, and $\bar{\beta} = (-g_1)^{|g_1|}$. Then $\alpha, \bar{\alpha}, \beta$, and $\bar{\beta}$ are irreducible in $\mathcal{B}(G)$. If $B = \alpha^u \bar{\alpha}^u \beta^v \bar{\beta}^v$, then the proofs of [7, Lemma 3.7 and Theorem 3.8] yield that $\rho(B^k) = \frac{kuM(G)+kv|g_1|}{2ku+2kv} = \frac{uM(G)+v|g_1|}{2u+2v} = \rho(B)$ for each $k \in \mathbb{N}$. Proceeding as in the first case, $\bar{\rho}(B) = \rho(B)$ and let $x = \frac{p}{q}$ be a rational with $\frac{|g_1|}{2} \le x \le \frac{M(G)}{2}$. This inequality forces both $2p- \mid g_1 \mid q$ and $M(G)q - 2p$ to be nonnegative (and not both zero). Setting $u = 2p- \mid g_1 \mid q$ and $v = M(G)q - 2p$, we obtain that $\rho(B) = x$. This completes the proof. □

We are able to completely determine the behavior of $\mathcal{B}(G)$ when G is an infinite abelian group.

Theorem 10. *If G is an infinite abelian group then $\overline{\mathcal{R}}(\mathcal{B}(G)) = \mathbb{Q} \cap [1, \infty)$ and hence $\mathcal{B}(G)$ is both fe and afe.*

Proof. That $\mathcal{B}(G)$ is fe follows directly from a Theorem of Kainrath [14]. We observe that if H is a subgroup of G, then $\overline{\mathcal{R}}(\mathcal{B}(H)) \subseteq \overline{\mathcal{R}}(\mathcal{B}(G))$ since any irreducible factors of a block from H must consist of elements of H. There are three cases to consider.

Case 1: G contains an element of infinite order. Then G contains a copy of $\mathbb{Z}$, so it suffices to show $\overline{\mathcal{R}}(\mathcal{B}(\mathbb{Z})) = \mathbb{Q} \cap [1, \infty)$. We will give an alternate argument that $\mathcal{B}(\mathbb{Z})$ is fe, which will lead to a proof for the asymptotic case.

Let $x = \frac{p}{q} \in [1, \infty)$, where $p, q \in \mathbb{N}$. We show that $\rho(B) = x$ for some $B \in \mathcal{B}(\mathbb{Z})$. First, if $x = 1$ then $\rho(\alpha) = 1 = x$ for any irreducible $\alpha \in \mathcal{A}(\mathcal{B}(\mathbb{Z}))$. Now suppose $x > 1$. Let $m, s, t \in \mathbb{N}$ such that $t \geq s$ and $m > 1$, and consider the block $B = (1^m \cdot (-m))^s((-1)^m \cdot m)^t$. The only possible irreducible divisors of B are $(1^m \cdot (-m))$, $((-1)^m \cdot m)$, $(1 \cdot (-1))$, and $(m \cdot (-m))$, which have lengths $m+1$ and 2. Since the given factorization of B contains only maximal length irreducible factors, it has minimal length so $l(B) = s+t$. Also, $B = (1 \cdot (-1))^{ms}((-m) \cdot m)^s((-1)^m m)^{t-s}$. Since this factorization of B contains the greatest possible number of irreducible factors of length 2, it has maximal length and $L(B) = ms + t$. Hence, $\rho(B) = \frac{ms+t}{s+t}$. Take $s = 1, t = 2q-1 \geq s$, and $m = 2p - 2q + 1 > 1$. Then $\rho(B) = \frac{(2p-2q+1)+(2q-1)}{1+(2q-1)} = \frac{2p}{2q} = x$. Now, for all $n \in \mathbb{N}$, $\rho(B^n) = \frac{mns+nt}{ns+nt} = \frac{ms+t}{s+t} = \rho(B)$, so by Lemma 2, $\overline{\rho}(B) = \rho(B)$.

Case 2: G contains elements of arbitrarily large finite order. Given $\frac{p}{q} \in \mathbb{Q} \cap [1, \infty)$, pick some $g \in G$ such that $\frac{|g|}{2} \geq \frac{p}{q}$. Since $\langle g \rangle$ is finite and cyclic, we know $M(\langle g \rangle) = D(\langle g \rangle) = |g|$ (see [5]). Thus, by our result for finite abelian groups, $\frac{p}{q} \in \overline{\mathcal{R}}(\mathcal{B}(\langle g \rangle)) \subseteq \overline{\mathcal{R}}(\mathcal{B}(G))$ and so asymptotic full elasticity holds in this case as well.

Case 3: G has a finite exponent. By Theorem 6 of Kaplansky [15], we know $G = \bigoplus_{i \in I} \mathbb{Z}_{n_i}$ for some I infinite and uniformly bounded family n_i of integers greater than 1. Using the Pigeonhole Principle and simple observations, we find that G has a subgroup of the form $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_p$ for some prime p. Without loss, we assume G is of this form and denote the standard basis for G by $\{e_i \,|\, i \in \mathbb{N}\}$.

First note that since $\mathbb{Z}_p$ is a subgroup of G, $\overline{R}(\mathbb{Z}_p) \subseteq \overline{R}(G)$. Since $D(\mathbb{Z}_p) = M(\mathbb{Z}_p)$, we have $[1, \frac{p}{2}] \subseteq \overline{R}(\mathbb{Z}_p)$ by Theorem 11.

Let $B_1, B_2, \ldots \in \mathcal{B}(G)$ be a sequence of blocks such that $\rho(B_i) < \rho(B_{i+1})$ for all $i \geq 1$ and $\lim_{i \to \infty} \rho(B_i) = \infty$. Since $\mathcal{B}(G)$ a $\mathcal{U}$-monoid, we may choose these B_i such that $\overline{\rho}(B_i) = \rho(B_i)$ for all $i \geq 1$. Indeed, for each i, there is an integer m_i such that $\overline{\rho}(B_i) = \rho(B_i^{m_i})$. But by Theorem 2, $\overline{\rho}(B_i^{m_i}) = \overline{\rho}(B_i) = \rho(B_i^{m_i})$, so we may choose a subsequence of $B_1^{m_1}, B_2^{m_2}, \ldots$ satisfying our desired conditions. Furthermore, with this choice we will have $L(B_i^k) = kL(B_i)$ and $l(B_i^k) = kl(B_i)$ by Theorem 2.

Let $x = \frac{a}{b} > \frac{p}{2}$ be a rational number. Pick B_i with $\overline{\rho}(B_i) = \rho(B_i) > x$. Since B_i involves only finite many elements of G, we may choose a finite subset $J \subset \mathbb{N}$ such that all the elements of B_i occur in the finite subgroup $H = \langle e_j \rangle_{j \in J}$. Pick $k \in \mathbb{N} \backslash J$ and set $C = e_k(-e_k)$. The independence of the basis elements assures that if B is an atom of $\mathcal{B}(G)$ dividing $B_i^t C^s$ for any positive integers t, s then either $H \cap B = \emptyset$ or $\langle e_k \rangle \cap B = \emptyset$. Thus $L(B_i^t C^s) = L(B_i^t) + L(C^s) = tL(B_i) + L(C^s)$ and $l(B_i^t C^s) = l(B_i^t) + l(C^s) = tl(B_i) + l(C^s)$. As before, we may choose an integer m such that $\overline{\rho}(C) = \rho(C^m)$ and observe that $L(C^{ms}) = mL(C^s)$ and $l(C^{ms}) = ml(C^s)$ for any positive integer s. However, in this case, we have an explicit choice for m. Since $\mathcal{U}(C) = \{C, e_k^p, -e_k^p\}$ it is clear that $m = p$ satisfies our conditions in this case. Thus for all t, s we have $\rho(B_i^t C^{ps}) = \frac{tL(B_i)+sL(C^p)}{tl(B_i)+sl(C^p)}$. In particular, with $s = bL(B_i) - al(B_i)$ and $t = a\, l(C^p) - bL(C^p)$ we obtain $\rho(B_i^t C^{ps}) = \frac{a}{b}$. Indeed, $s > 0$ since $\rho(B_i) > \frac{a}{b}$ by assumption and $t > 0$ since $\rho(C^p) = \frac{p}{2} < \frac{a}{b}$. Furthermore, for all n $L((B_i^t C^{ps})^n) = ntL(B_i) + nsL(C^p)$ and $l((B_i^t C^{ps})^n) = ntl(B_i) + nsl(C^p)$ so $\overline{\rho}(B_i^t C^{ps}) = \rho(B_i^t C^{ps})$, completing the argument. □

If M is a Krull monoid with divisor class group G such that each divisor class contains a prime divisor, then $\mathcal{L}(M) = \mathcal{L}(\mathcal{B}(G))$. Hence, Theorems 8 and 9 are applicable to such Krull monoids with the correct conditions applied when G is finite.

References

[1] D. D. Anderson, D. F. Anderson, S. T. Chapman and W. W. Smith, Rational elasticity of factorizations in Krull domains, *Proc. Amer. Math. Soc.* **117**(1993), 37–43.
[2] D. F. Anderson, Elasticity of factorizations in integral domains: a survey, *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, **189**(1997), 1–29.
[3] D. F. Anderson, and P. Pruis, Length functions on integral domains, *Proc. Amer. Math. Soc.* **111**(1991), 933–937.
[4] M. Banister, J. Chaika, S. T. Chapman and W. Meyerson, On the arithmetic of arithmetical congruence monoids, submitted for publication.
[5] S. T. Chapman, On the Davenport constant, the cross number, and their application in factorization theory, *Zero-dimensional commutative rings (Knoxville, TN, 1994)*, Marcel Dekker, New York, 1997, 167–190.
[6] S. Chapman and A. Geroldinger, Krull domains and monoids, their sets of length and associated combinatorial problems, *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, **189**(1997), 73-112.
[7] S. T. Chapman, M. Holden and T. Moore, Full elasticity in atomic monoids and integral domains, to appear in *Rocky Mountain J. Math.*.
[8] S. T. Chapman and B. McClain, Irreducible polynomials and full elasticity in rings of integer-valued polynomials, to appear in *J. Algebra*.
[9] S. T. Chapman and J. C. Rosales, On the asymptotic values of length functions in Krull and finitely generated commutative monoids, *J. Aust. Math. Soc.* **74**(2003), 421–436.
[10] A. Geroldinger and F. Halter-Koch, On the asymptotic behaviour of lengths of factorizations, *J. Pure Appl. Algebra* **77**(1992), 239–252.
[11] A. Geroldinger and F. Halter-Koch, Transfer principles in the theory of non-unique factorizations, *Lecture Notes in Pure and Applied Mathematics* **241**(2005), 114–141.
[12] A. Geroldinger and F. Halter-Koch, Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory, Chapman & Hall/CRC, New York, 2005.
[13] F. Halter–Koch, Elasticity of factorizations in atomic monoids and integral domains, *J. Theorie des Nombres Bordeaux*, **7** (1995), 367–385.
[14] F. Kainrath, Factorization in Krull monoids with infinite class group, *Colloq. Math.* **80**(1999), 23–30.
[15] I. Kaplansky, Infinite Abelian Groups, third printing, The University of Michigan Press, 1960.

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