A NEW CHARACTERIZATION OF HALF-FACTORIAL KRULL MONOIDS

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Abstract. Let M be a Krull monoid. Then every element of M may be written as a finite product of irreducible elements. If for every $a \in M$, each two factorizations of a have the same number of irreducible elements, then M is called half-factorial. Using a property of element exponentiation, we provide a new characterization of half-factoriality, valid for all Krull monoids whose class group has torsion-free rank at most one.

1. Introduction and Main Result

The arithmetic of Krull monoids has attracted a lot of attention in recent years [4, 12, 13, 15], and the present paper is devoted to that topic. Recall that an integral domain is a Krull domain if and only if its multiplicative monoid of nonzero elements is a Krull monoid, and a noetherian domain is Krull if and only if it is integrally closed. A more detailed discussion of Krull monoids will be given in Section 2. Half-factoriality is one of the most classic properties of factorization theory (see the surveys [5, 6] and [20, 23] for recent progress on half-factorial domains). We recall some basic definitions which allow us to formulate the main result of the present paper.

Suppose that M is a Krull monoid with class group G, $G_P \subseteq G$ the set of classes containing prime divisors, and let $a \in M$. Then a has a factorization $a = u_1 \cdot \ldots \cdot u_k$ into irreducible elements (atoms) $u_1, \ldots, u_k \in M$. The number of factors, k, is called the length of the factorization. The set $\mathsf{L}(a)$ of all possible factorization lengths is the set of lengths of a, and $\rho(a) = \sup \mathsf{L}(a) / \min \mathsf{L}(a)$ is called the elasticity of a. The monoid M is said to be half-factorial if $|\mathsf{L}(a)| = 1$ for all $a \in M$ (in this case we also say that the set G_P is half-factorial). Suppose that every divisor class of G contains a prime divisor. Then it is classic that M is half-factorial if and only if $|G| \leq 2$. This is far from being true in general. On the contrary, it is an open conjecture that for every abelian group A there is a half-factorial Krull monoid with class group isomorphic to A (see [14, Proposition 3.7.9] and [11]). Moreover, even in the case of a finite class group, the maximal size and the structure of half-factorial sets is understood only in very special cases [19, 24, 25, 26, 27].

The main aim of the present paper is to establish a new characterization of half-factoriality which is valid for a large class of Krull monoids.

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Theorem 1.1. *Let M be a Krull monoid, $\varphi\colon M \to F$ a cofinal divisor homomorphism into a free monoid F and $G = \mathcal{C}(\varphi)$ is class group. If the torsion-free rank of G is at most 1, then the following statements are equivalent:*

(a) *M is half-factorial.*
(b) *We have $\rho(a) = \rho(a^m)$ for all $a \in M$ and all $m \in \mathbb{N}$.*

Property (*b*) of the above theorem was recently studied in the context of finitely generated monoids [8], and closely related concepts were investigated in [2]. In Section 3 we will provide an example showing that the above characterization no longer holds if the torsion-free rank of the class group is larger than one.

In the development of this manuscript, we communicated several ideas and early results with P. García-Sánchez and J. Rosales [8, 9], who gave helpful feedback and discovered a counterexample for (*b*) ⇒ (*a*) in the setting of finitely generated monoids [8, Example 1]. Those authors then independently pursued a characterization of (*b*) for finitely generated monoids. Our theorems, obtained independently, concern the question of half-factoriality in Krull monoids.

2. Preliminaries

Our notation and terminology are consistent with [14]. We briefly gather some key notions.

Monoids and Factorizations. By a *monoid*, we mean a commutative, cancellative semigroup with unit element. Let M be a monoid. We denote by $\mathcal{A}(M)$ the set of irreducible elements (atoms) of M, by $M^\times$ the group of invertible elements, and by $M_{\rm red} = \{aM^\times \mid a \in M\}$ the associated reduced monoid of M. We denote by $\mathsf{q}(M)$ a quotient group of M with $M \subseteq \mathsf{q}(M)$. For a set P, we denote by $\mathcal{F}(P)$ the *free* (*abelian*) *monoid* with basis P. Then every $a \in \mathcal{F}(P)$ has a unique representation in the form

$$a = \prod_{p\in P} p^{\mathsf{v}_p(a)} \quad \text{with} \quad \mathsf{v}_p(a) \in \mathbb{N}_0 \quad \text{and} \quad \mathsf{v}_p(a) = 0 \quad \text{for almost all} \quad p \in P\,.$$

We call $|a| = \sum_{p\in P} \mathsf{v}_p(a)$ the *length* of a. The free monoid $\mathsf{Z}(M) = \mathcal{F}\big(\mathcal{A}(M_{\rm red})\big)$ is called the *factorization monoid* of M, and the unique homomorphism

$$\pi\colon \mathsf{Z}(M) \to M_{\rm red} \quad \text{satisfying} \quad \pi(u) = u \quad \text{for each} \quad u \in \mathcal{A}(M_{\rm red})$$

is called the *factorization homomorphism* of M. For $a \in M$, we set

$$\mathsf{Z}_M(a) = \mathsf{Z}(a) = \pi^{-1}(aM^\times) \subseteq \mathsf{Z}(M) \quad \text{is the } \textit{set of factorizations} \text{ of } a\,,$$
$$\mathsf{L}_M(a) = \mathsf{L}(a) = \big\{|z| \;\big|\; z \in \mathsf{Z}(a)\big\} \subseteq \mathbb{N}_0 \quad \text{is the } \textit{set of lengths} \text{ of } a,$$
$$L(a) = \sup \mathsf{L}(a) \in \mathbb{N} \cup \{\infty\}, \quad \ell(a) = \min \mathsf{L}(a), \quad \text{and if } a \notin M^\times, \text{ then}$$
$$\rho(a) = L(a)/\ell(a) \quad \text{is the } \textit{elasticity} \text{ of } a\,.$$

Note that $\mathsf{L}(a) = \{0\}$ if and only if $a \in M^\times$. We define the *elasticity* of M to be $\rho(M) = \sup\{\rho(x) \,|\, x \in M \backslash M^\times\}$. M has *accepted elasticity* if $\rho(M) < \infty$ and there exists $a \in M$ with $\rho(a) = \rho(M)$. We say that M is *atomic* if $|Z(a)| \geq 1$ for all $a \in M$. If $N \subseteq M$ are reduced monoids, then N is *divisor closed* in M if whenever $a \in N$ and $b, c \in M$ with $a = bc$, then $b, c \in N$.

Let G be an additive abelian group and $G_0 \subseteq G$ a subset. Then $[G_0] \subseteq G$ denotes the submonoid generated by G_0 and $\langle G_0 \rangle \subseteq G_0$ denotes the subgroup generated by G_0. Set $\mathrm{Tor}(G)$ to be the torsion elements of G. A family $(e_i)_{i\in I}$ of elements of G is said to be *independent* if they are independent in G as a $\mathbb{Z}$-module, i.e. if $e_i \neq 0$ for all $i \in I$ and, for every family $(m_i)_{i\in I} \in \mathbb{Z}^{(I)}$,

$$\sum_{i\in I} m_i e_i = 0 \qquad \text{implies} \qquad m_i e_i = 0 \quad \text{for all} \quad i \in I \,.$$

The *torsion-free rank* $\mathsf r_0(G)$ of G is the cardinality of a maximal independent family in $G/\mathrm{Tor}(G)$. Thus G is a torsion group if and only if $\mathsf r_0(G) = 0$.

Krull monoids. The theory of Krull monoids is presented in the monographs [14, 17, 18]. We briefly summarize what is needed in the sequel. Let M and D be monoids. A monoid homomorphism $\varphi \colon M \to D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies $a \,|\, b$, for all $a, b \in M$.
- *cofinal* if, for every $a \in D$, there exists some $u \in M$ such that $a \,|\, \varphi(u)$.
- a *transfer homomorphism* if $\varphi(M)D^\times = D$, $\varphi^{-1}(D^\times) = M^\times$, and whenever $\varphi(m) = bc$, there exist $v, w \in M$ and $e \in D^\times$ such that $m = vw$, $\varphi(v) = eb$, and $\varphi(w) = e^{-1}c$.

We call $\mathcal C(\varphi) = \mathsf q(D)/\mathsf q(\varphi(M))$ the class group of φ and use additive notation for this group. For $a \in \mathsf q(D)$, we denote by $[a] = [a]_\varphi = a\,\mathsf q(\varphi(M)) \in \mathsf q(D)/\mathsf q(\varphi(M))$ the class containing a. If $\varphi \colon M \to \mathcal F(P)$ is a cofinal divisor homomorphism, then

$$G_P = \{[p] = p\mathsf q(\varphi(M)) \mid p \in P\} \subseteq \mathcal C(\varphi)$$

is called the *set of classes containing prime divisors*, and we have $[G_P] = \mathcal C(\varphi)$. If $M \subseteq D$ is a submonoid, then M is called *cofinal* (resp., *saturated*) in D if the imbedding $M \hookrightarrow D$ is cofinal (resp., a divisor homomorphism).

The monoid M is called a *Krull monoid* if it satisfies one of the following equivalent conditions [14, Theorem 2.4.8]:

- M is v-noetherian and completely integrally closed.
- M_{red} is a saturated submonoid of a free monoid.

In particular, M is a Krull monoid if and only if M_{red} is a Krull monoid. A domain is a Krull domain if and only if its multiplicative monoid is a Krull monoid, and a noetherian domain is Krull if and only if it is integrally closed. Regular congruence monoids in Krull domains are Krull [14, Proposition 2.11.6]. Monoid domains and power series domains that are Krull are discussed in [16, 21, 22]. For the role of Krull monoids in module theory see [7]. The arithmetic of Krull monoids is studied via transfer homomorphisms and associated block monoids. We recall the required terminology and collect the results needed for the sequel.

Monoids of zero-sum sequences and block monoids. Let G be an additive abelian group, $G_0 \subseteq G$ a subset and $\mathcal F(G_0)$ the free monoid with basis G_0. In the tradition of combinatorial number theory, the elements of $\mathcal F(G_0)$ are called *sequences* over G_0. Thus a sequence $S \in \mathcal F(G_0)$ will be written in the form

$$S = \prod_{g\in G_0} g^{\mathsf v_g(S)} \,,$$

and we use all the notions (such as the length) as in general free monoids. If S is not the unit element, then we can write $S = g_1 \cdot \ldots \cdot g_l$ and tacitly assume $l \in \mathbb N$ and $g_1, \ldots, g_l \in G_0$. We call $\mathrm{Supp}(S) = \{g \in G_0 \,|\, \mathsf v_g(S) > 0\}$ the *support* of S and $\sigma(S) = \sum_{g\in G_0} \mathsf v_g(S) g$ the *sum* of S, which is an element of G. The monoid

$$\mathcal B(G_0) = \{S \in \mathcal F(G_0) \mid \sigma(S) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0, and its elements are called *zero-sum sequences* over G_0. We suppress mention of the group G, since the same monoid $\mathcal{B}(G_0)$ is attained no matter which group G extending $\langle G_0 \rangle$ we are working inside. We shall freely use the following fact without further mention:

Fact 2.1. If $S' \subseteq S \subseteq G$, then $\mathcal{B}(S')$ is a divisor-closed submonoid of $\mathcal{B}(S)$. [14, Prop. 2.5.6]

For every arithmetical invariant $*(M)$ defined for a monoid M, we write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$. In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$. Clearly, $\mathcal{B}(G_0) \subseteq \mathcal{F}(G_0)$ is saturated, and hence $\mathcal{B}(G_0)$ is a Krull monoid.

We define the *Davenport constant* of G_0 by

$$\mathsf{D}(G_0) = \sup\big\{ |U| \; \big| \; U \in \mathcal{A}(G_0)\big\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$

which is a central invariant in zero-sum theory (see [10] for its relevance in factorization theory).

Fact 2.2. If G_0 is finite, then $\mathsf{D}(G_0) < \infty$ and $\mathcal{B}(G_0)$ is finitely generated [14, Theorem 3.4.2].

We will make substantial use of the following result [14, Section 3.4] relating Krull monoids to associated block monoids.

Lemma 2.3. *Let M be a Krull monoid, $\varphi\colon M \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, and $G_P \subseteq G$ the set of classes containing prime divisors. Let $\widetilde{\boldsymbol{\beta}}\colon F \to \mathcal{F}(G_P)$ denoted the unique homomorphism defined by $\widetilde{\boldsymbol{\beta}}(p) = [p]$ for all $p \in P$. Then homomorphism $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \circ \varphi\colon M \to \mathcal{B}(G_P)$ is a transfer homomorphism. In particular, we have*

1. $\mathsf{L}_M(a) = \mathsf{L}_{\mathcal{B}(G_P)}\big(\boldsymbol{\beta}(a)\big)$ *and hence* $\rho(a) = \rho\big(\boldsymbol{\beta}(a)\big)$ *for all* $a \in M$.
2. *M is half-factorial if and only if $\mathcal{B}(G_P)$ is half-factorial.*

The homomorphism $\boldsymbol{\beta}$ is called the *block homomorphism*, and $\mathcal{B}(G_P)$ is called the *block monoid* associated to φ.

3. Technical results

The current manuscript will concentrate on a concept called strong tautness, related to element exponentiation.

Definition 3.1. An atomic monoid M is **taut** if for every nonunit $x \in M$, $\rho(x) < \infty$ and there is an $n \in \mathbb{N}$ such that for all $m \geq n$, $\rho(x^n) = \rho(x^m)$. M is **strongly taut** if in the definition of taut we can universally take $n = 1$.

Many monoids are taut, including all finitely-generated monoids [1, Thm 12] and, more generally, all monoids M such that for all nonunits $x \in M$, the divisor closure $[[x]]$ of $\{x\}$ is finitely generated. The hypothesis of tautness allows one to conclude many strong factorization properties of the monoid; as an example, see [2, 3]. Because $\rho(x) = 1$ for all its elements, a half-factorial monoid is trivially taut and strongly taut. Over block monoids, both half-factoriality and strong tautness have *finite character*. That is to say, the block monoid $\mathcal{B}(S)$ is half-factorial (resp. strongly taut) iff each $\mathcal{B}(S')$ is half-factorial (resp. strongly taut) for all finite $S' \subseteq S$. Indeed, verifying these properties in $\mathcal{B}(S)$ involves inspecting the factorizations of each zero-sum sequence B; the factors of B are all elements of the divisor closed submonoid $\mathcal{B}(\mathrm{Supp}(B))$ (by Fact 2.1) and $\mathrm{Supp}(B)$ is a finite subset of S. Thus to prove the equivalence of strong tautness and half-factoriality, it suffices to verify the equivalence for all finite $S' \subseteq S$. In the block monoid, the trivial zero-sum sequence 0 is a prime element, and thus must appear in every factorization. Hence, it has no effect on factorization properties such as half-factoriality or strong tautness (see [2] for more details). For the sequel, we shall safely assume $0 \notin S$.

In this article, we investigate the relationship between strong tautness and half-factoriality in Krull monoids. We prove that for (a generalization of) Krull monoids with class group of torsion-free rank at most 1, the conditions are equivalent, so that strong tautness provides a characterization of half-factoriality for such monoids. For Krull monoids whose class group is torsion, strong tautness thus yields an alternate characterization of the old and well-known characterization of half-factoriality involving the semilength function known as the *cross number* (see [14, Proposition 6.7.3]). When the class group has torsion-free rank 2 or greater, the properties need not be equivalent, as demonstrated by Example 3.8.

We now collect several simple observations about element exponentiation in general monoids, as well as basic results about block monoids over certain groups of torsion-free rank at most 1.

Proposition 3.2. *Let M be an atomic monoid, $n \in \mathbb{N}$ and $x \in M$ a nonunit with $\rho(x)$ finite. If $\rho(x^n) = \rho(x)$, then $L(x^n) = nL(x)$ and $l(x^n) = nl(x)$.*

Proof. Since $nL(x) \leq L(x^n)$ and $nl(x) \geq l(x^n)$, if either inequality were strict, we would have $\rho(x^n) > \rho(x)$. □

Proposition 3.3. *If M is an atomic monoid with accepted elasticity and x is any element of M for which $\rho(x) = \rho(M)$, then*

1. *$\rho(x^n) = \rho(x)$ for all $n \in \mathbb{N}$*
2. *if $a \in \mathcal{A}(M)$ appears in both a longest and a shortest factorization of x, then $\rho(M) = \rho(x) = 1$, i.e. M is half-factorial.*

Proof. The first statement is immediate: $\rho(x) \leq \rho(x^n) \leq \rho(M) = \rho(x)$. For the second statement, let such an a be given and consider $x/a \in M$. If $x = a$, then $\rho(M) = \rho(x) = 1$. Otherwise, we have $L(x/a) = L(x) - 1$ and $\ell(x/a) = \ell(x) - 1$, so if $\rho(x) > 1$ then $\rho(x/a) > \rho(x) = \rho(M)$, a contradiction. □

Remark 3.4. Let S be a finite subset of an abelian group of torsion-free rank at most 1. Then $\langle S \rangle$ is a finitely-generated abelian group of torsion-free rank at most 1, hence $\langle S \rangle$ is isomorphic to either a finite abelian group H or the direct product of $\mathbb{Z}$ and H, for some finite abelian group H. Without loss of generality in proofs, when dealing with a fixed finite subset S, we shall assume that $S \subseteq \mathbb{Z} \oplus H$.

Definition 3.5. If M is an atomic monoid, then $a \in M$ is **absolutely irreducible** if whenever $b \in \mathcal{A}(M)$ and $b \mid a^n$ for some $n \in \mathbb{N}$, then a and b are associates.

Note that since M is atomic, any absolute irreducible must be irreducible. There are several equivalent statements for a being absolutely irreducible, for example, that the divisor-closed submonoid $[[a]]$ is a unique factorization monoid. For further general properties, see [14, Definition 7.1.3]. In the context of block monoids, absolute irreducibility is equivalent to isolating a minimal support set.

Proposition 3.6. *Let G be an abelian group and $S \subseteq G$. Then $B \in \mathcal{A}(\mathcal{B}(S))$ is absolutely irreducible iff for all $B' \in \mathcal{A}(\mathcal{B}(S))$ with Supp$(B') \subseteq$ Supp(B), we have $B' = B$.*

Proof. Since block monoids are reduced, we do not need to be concerned with associates. If $B, B' \in \mathcal{A}(\mathcal{B}(S))$ with $\mathrm{Supp}(B') \subseteq \mathrm{Supp}(B)$, then there is some power B^n of B such that $B'|B^n$. Conversely, since $\mathcal{B}(\mathrm{Supp}(B))$ is divisor-closed in $\mathcal{B}(S)$ (Fact 2.1), any factor B' of some B^n must be an element of $\mathcal{B}(\mathrm{Supp}(B))$, i.e. $\mathrm{Supp}(B') \subseteq \mathrm{Supp}(B)$. Thus absolute irreducibility of B is equivalent to the stated property about supports. □

In a torsion group G, we immediately conclude by the previous proposition that the absolute irreducibles are the zero-sum sequences $g^{\mathrm{ord}_G(g)}$ for $g \in S$. For groups of torsion-free rank 1, the absolute irreducibles are also easily classified.

Lemma 3.7. *Let G is a group of torsion-free rank at most 1 and $S \subseteq G$. The absolute irreducibles of $\mathcal{B}(S)$ are exactly the zero-sum sequences of the form:*

1. *$g^{\mathrm{ord}_G(g)}$ for some $g \in S \cap$ Tor(G), or*
2. *B an irreducible with $|$Supp$(B)| = 2$ and Supp$(B) \subseteq S \setminus$ Tor(G).*

Furthermore, for any zero-sum sequence $A \in \mathcal{B}(S)$, there is an $n \in \mathbb{N}$ such that A^n factors as a product of absolute irreducibles.

Proof. As already mentioned, elements of the first kind are absolutely irreducible. By the minimality of support (Proposition 3.6), any other absolute irreducibles in $\mathcal{B}(S)$ must have support contained in $G\setminus\mathrm{Tor}(G)$.

Now suppose that $B = g^n h^m$ is an irreducible where g and h are elements of S of infinite order. The support of B is clearly minimal because of the infinite order of g and h. B is also the unique irreducible with $\{g, h\}$ as its support. Indeed, let $A = g^i h^j$ be irreducible for some $i, j \in \mathbb{N}$. Since B is irreducible, either $i \geq n$ or $j \geq m$; without loss of generality, assume the former. Since A is also irreducible, $j \leq m$. Since $n \cdot g = m \cdot h$ and $i \cdot g = j \cdot h$, we get $mi \cdot h = ni \cdot g = nj \cdot h$, and because h is not torsion, $mi = nj$. Thus $1 \leq m/j = n/i \leq 1$, so $m = j$ and $n = i$ and B is absolutely irreducible by Proposition 3.6.

We now shall show that any other $A \in \mathcal{A}(\mathcal{B}(S))$ does not have minimal support (under inclusion) and so cannot be absolutely irreducible. This will fall out of our proof of the second statement of the lemma, namely that any zero-sum sequence $A \in \mathcal{B}(S)$ has a power which factors as a product of absolute irreducibles.

We proceed by induction on the cardinality of $\mathrm{Supp}(A)$. Since $\mathrm{Supp}(A)$ is finite, we may assume by Remark 3.4 that $\mathrm{Supp}(A) \subseteq \mathbb{Z} \oplus H$ for some finite group H. Set $e = \exp(H)$. Then $A^e = A_1 A_2$, where $\mathrm{Supp}(A_1) \subseteq H$ and $\mathrm{Supp}(A_2) \cap H = \emptyset$. Clearly A_1 factors as a product of the absolute irreducibles $g^{\mathrm{ord}(g)}$ for $g \in \mathrm{Supp}(A_1)$, so we are left with the case where $\mathrm{Supp}(A)$ contains no torsion elements (and thus $|\mathrm{Supp}(A)| \geq 2$). We shall show that a power of A factors as a product of absolute irreducibles of the second kind. If $|\mathrm{Supp}(A)| = 2$, then A is necessarily a power of the unique absolute irreducible on that support. Assume now, $A = (m_1 + g_1)^{a_1} \cdots (m_k + g_k)^{a_k} (-n_1 + h_1)^{b_1} \cdots (-n_l + h_l)^{b_l}$, where each $m_i, n_j \in \mathbb{N}$, $g_i, h_j \in H$, $a_i, b_i \geq 1$ and $k + l > 2$.

If $a_k m_k \geq b_l n_l$, set $x = m_k e$. The zero-sum sequence $B = (m_k + g_k)^{b_l e n_l} (-n_l + h_l)^{b_l x}$ is a power of the absolute irreducible with support $\{m_k + g_k, -n_l + h_l\}$. Furthermore, B divides A^x. Indeed $\mathsf{v}_{m_k+g_k}(A^x) = a_k x > b_l e n_l = \mathsf{v}_{m_k+g_k}$ and

$\mathsf{v}_{-n_l+h_l}(A^x) = b_l x = \mathsf{v}_{-n_l+h_l}(B)$. Since $-n_l + h_l$ is not in the support of A^x/B, A does not have minimal support and thus is not absolutely irreducible by Proposition 3.6. Also, by the induction hypothesis, there is a nonzero $y \in \mathbb{N}$ such that $(A^x/B)^y$ factors as absolute irreducibles of the second kind. Thus xy is our desired exponent for A. If $a_k m_k \le b_l n_l$, then an analogous proof applies with $x = n_l e$. □

We shall eventually show that absolute irreducibles play a critical role in determining half-factoriality for restricted block monoids over groups of torsion-free rank ≤ 1. As a contrast, we present the following example of Schmid [28] to show that there are strongly taut block monoids over groups of torsion-free rank 2 or more which are not half factorial. This example was adapted from an example of García-Sánchez and Rosales for finitely generated monoids [8, Example 1].

Proposition 3.8. *Let G be an abelian group of torsion-free rank 2 or more and let $e_1, e_2 \in G$ be two independent elements of infinite order. Set $e_0 = e_1 + e_2$. Let $S = \{e_0, e_1, e_2, -e_0, -e_1, -e_2\}$. Then $\mathcal{B}(S)$ is strongly taut but not half factorial.*

Proof. For $i \in \{0, 1, 2\}$, set $A_i = (-e_i)e_i$, which are clearly absolute irreducibles by Proposition 3.6. Set $U = (-e_0)e_1e_2$ and $-U = e_0(-e_1)(-e_2)$, which are also absolutely irreducible and set $W = (-U)U = A_0A_1A_2$. These irreducible factorizations have different lengths, so $\mathcal{B}(S)$ is not half factorial. Note that $A_0, A_1, A_2, U, -U$ are the only irreducibles of $\mathcal{B}(S)$. From this we conclude that the longest factorization of W^n has length $3n$ and the shortest factorization of W^n has length $2n$, so $\rho(W^n) = \frac{3}{2}$ for all $n \in \mathbb{N}$.

Now let $B \in \mathcal{B}(S)$ be an arbitrary zero-sum sequence and write $\mathsf{v}_s(B)$ for the multiplicity of a given $s \in S$ in B. There is a maximal $n \in \mathbb{N}_0$ such that W^n divides B, call it $w(B)$. We claim that $L(B) = 3w(B) + r(B)$ and $l(B) = 2w(B) + r(B)$, where

$$r(B) = \mathsf{v}_{e_0}(B) + \mathsf{v}_{e_1}(B) + \mathsf{v}_{e_2}(B) - 3w(B).$$

We prove this by induction on $r(B)$. We see that $r(B) = 0$ iff $B = W^n$ for some $n \in \mathbb{N}$ and in this case we have already shown the length computations to hold. Suppose $r(B) > 0$, so that $B/W^{w(B)}$ is nontrivial. Suppose $U|B/W^{w(B)}$, so that $\mathsf{v}_{e_0}(B), \mathsf{v}_{-e_1}(B), \mathsf{v}_{-e_2}(B) > w(B)$. We claim that U must appear in every factorization of B. Assume otherwise, i.e., there is some factorization F of B in which no U appears. Considering supports, we must then have $\mathsf{v}_{e_0}(B)$ copies of A_0, $\mathsf{v}_{-e_1}(B)$ copies of A_1, and $\mathsf{v}_{-e_2}(B)$ copies of A_2 in F. But then we have $\min\{\mathsf{v}_{e_0}(B), \mathsf{v}_{-e_1}(B), \mathsf{v}_{-e_2}(B)\} > w(B)$ copies of W dividing B, a contradiction. Hence U appears in every factorization of B and thus $L(B) = L(B/U) + 1$ and $l(B) = l(B/U)+1$. Since $U|B/W^{w(B)}$, we conclude $w(B/U) = w(B)$, so $r(B/U) = r(B) - 1$ and our induction hypothesis applies. Analogous arguments hold if any of the other irreducibles divided $B/W^{w(B)}$.

Given this knowledge about longest and shortest factorization lengths, we immediately conclude strong tautness. For $w(B^k) = kw(B)$ and so $r(B^k) = kr(B)$. Thus $\rho(B^k) = \rho(B)$. □

Note that in this example, all the irreducibles are absolutely irreducible. The major difference between this example and rank ≤ 1 groups is the size of the support. For a group of rank ≤ 1, for any $s \in S$, all the absolute irreducibles in which s appears have the same cardinality for their support. Namely, if s is torsion, the unique absolute irreducible in which s appears has singleton support; if

s is infinite order, then it may appear in the support of many absolute irreducibles, but all these absolute irreducibles have support of size 2. On the other hand, in this example, every $s \in S$ appears in the support of an absolute irreducible of size 2 and one of size 3. It could be an area of future research to examine whether this property of locally uniform support size is necessary for half-factoriality.

4. Main Result

We now can describe the configurations that control the behavior of $\mathcal{B}(S)$ with respect to half factoriality and strong tautness. By a *semigroup*, we mean a set with a binary operation which is associative but need not have identity.

Definition 4.1. Let $S \subseteq G$, where G is a group of torsion-free rank 1. S is **nice** if S can be enumerated as $\{s_1, s_2, s_3, s_4\}$, with the s_i all distinct and of infinite order, such that the semigroup generated by S contains 0, but the two semigroups generated by $\{s_1, s_2\}$ and $\{s_3, s_4\}$ do not contain 0.

Intuitively, nice subsets correspond to those S which consist of two "positive" elements and two "negative" elements. In more detail, if S is nice, then since S is finite, we may assume by Remark 3.4 that $S \subseteq \mathbb{Z} \oplus H$, for some finite abelian group H. We then find that $S = \{m_2 + g_2, m_1 + g_1, -n_1 + h_1, -n_2 + h_2\}$, for some $m_2, m_1, n_1, n_2 \in \mathbb{N}$ and $g_2, g_1, h_1, h_2 \in H$.

We shall now show that half-factoriality and strong tautness are equivalent for nice subsets of groups G of torsion-free rank 1.

Lemma 4.2. *Let S be a nice subset of a group G of torsion-free rank 1. Then there exists a zero-sum sequence $B \in \mathcal{B}(S)$ with Supp$(B) = S$ and $\rho(B) = 1$.*

Proof. After identification, we may assume without loss of generality that $S = \{m_2+g_2, m_1+g_1, -n_1+h_1, -n_2+h_2\} \subseteq \mathbb{Z}\oplus H$ for some finite abelian group H. Let A_1, A_2 be the (unique) absolute irreducibles having support $\{m_1 + g_1, -n_1 + h_1\}$ and $\{m_2 + g_2, -n_2 + h_2\}$, respectively. Then $\mathrm{Supp}(A_1A_2) = S$ and A_1A_2 has a factorization of length 2. Consider, then, the nonempty set of zero-sum sequences B such that $\ell(B) = 2$ and $\mathrm{Supp}(B) = S$. This set is partially ordered by the "divides" relation, namely, for A and B in the set, $A \leq B$ iff $A|B$. Since $\mathcal{B}(S)$ is a finitely generated atomic monoid (Fact 2.2), we may choose a zero-sum sequence B in the set which is minimal under this partial order.

Consider any other factorization $B_1 \cdots B_k$ of B into irreducibles. If some B_i has three or more elements in its support, then we may choose an irreducible factor B_j with $j \neq i$ such that $\mathrm{Supp}(B_iB_j) = S$. But then $\mathrm{Supp}(B_iB_j) = S$ and B_iB_j factors as the product of two irreducibles, so by minimality, $B = B_iB_j$ and $k = 2$. The remaining case is when all the B_i have exactly two elements in their support, one of the form $m_i + g_i$ and one of the form $-n_j + h_j$. Since $\mathrm{Supp}(B) = S$, a simple inspection of cases shows that two of the irreducibles B_i and B_j together will have $\mathrm{Supp}(B_iB_j) = S$, so by minimality $B = B_iB_j$. Thus all factorizations of B have length 2 and $\rho(B) = 1$. □

Theorem 4.3. *If $S \subseteq G$ is nice, then $\mathcal{B}(S)$ is strongly taut iff $\mathcal{B}(S)$ is half factorial.*

Proof. For the nontrivial direction, suppose $\mathcal{B}(S)$ is strongly taut and let B be the zero-sum sequence guaranteed by Lemma 4.2. Let A be any element of $\mathcal{B}(S)$. Since $\mathrm{Supp}(B) = S$, we may choose $r \in \mathbb{N}$ such that $A|B^r$. Then B^{r+1}/A is a nontrivial zero-sum sequence and we may factor it into x irreducibles, for some $x \geq 1$. By strong tautness, $1 = \rho(B) = \rho(B^{r+1}) \geq \frac{x+L(A)}{x+\ell(A)}$, so $L(A) = \ell(A)$. □

Theorem 4.4. *Suppose G is a group of torsion-free rank 0 or 1 and $S \subseteq G$. Then the following are equivalent:*

1. *$\mathcal{B}(S)$ is half factorial.*
2. *$\mathcal{B}(S)$ is strongy taut.*
3. *For every nice $T \subseteq S$, $\mathcal{B}(T)$ is strongly taut and for every irreducible $a \in \mathcal{A}(\mathcal{B}(S))$, $\rho(a^n) = \rho(a) = 1$ for all n.*
4. *For every nice $T \subseteq S$, $\mathcal{B}(T)$ is half factorial and for every irreducible $a \in \mathcal{A}(\mathcal{B}(S))$, $\rho(a^n) = \rho(a) = 1$ for all n.*

Proof. (1) $\Rightarrow$ (2) follows from half-factoriality implying strong tautness, while (2) $\Rightarrow$ (3) follows from the definition and finite character of strong tautness. Conditions (3) and (4) are equivalent by Theorem 4.3.

For (3) $\Rightarrow$ (1), suppose (3) (and thus (4)). By the finite character of half-factoriality, we reduce to showing that $\mathcal{B}(S')$ is half factorial for every finite $S' \subseteq S$. Let some such S' be given; assume . We may assume $0 \notin S'$ and that S' has no extraneous elements.

Since S' is a finite subset of an abelian group of torsion-free rank at most 1, by Remark 3.4 we may assume that $S' \subseteq \mathbb{Z} \oplus H$, where H is a finite abelian group. By Fact 2.2, $\mathcal{B}(S')$ is finitely-generated and thus has accepted elasticity by [1, Thm 7].

Assume $\rho(S') > 1$ and fix $B \in \mathcal{B}(S')$ such that $\rho(B) = \frac{a}{b} = \rho(S')$. Then $L(B) = aR$ and $\ell(B) = bR$ for some positive integer R, and we can obtain irreducible factorizations $B = \alpha_1 \cdots \alpha_{aR} = \beta_1 \cdots \beta_{bR}$. By Lemma 3.7 there is a positive integer w such that, simultaneously, each of the α_i^w and β_j^w factors as a product of absolute irreducibles. Since $\rho(a^w) = 1$ for all irreducibles a, each α_i^w and β_j^w factors as exactly w irreducibles by Proposition 3.2. On the other hand, Proposition 3.3 implies $\rho(B^w) = \rho(B)$, so again by Proposition 3.2, $L(B^w) = wL(B)$ and $\ell(B^w) = w\ell(B)$. Thus we obtain factorizations:

$$\prod_{i=1}^{aR} \prod_{j=1}^{w} P_{i,j} = B^w = \prod_{i=1}^{bR} \prod_{j=1}^{w} Q_{i,j} \tag{4.1}$$

where the factorization on the left is a longest factorization of B^w, the factorization on the right is a shortest, and each $P_{i,j}$ and $Q_{i,j}$ is absolutely irreducible.

We claim that $\mathrm{Supp}(B) \cap H$ is empty. For if $h \in H$ were in the support, then $h^{\mathrm{ord}_H(h)}$ would have to appear in any factorization of B^x into absolute irreducibles– in particular the longest and shortest factorizations in equation 4.1. Twice applying Proposition 3.3, we see $\rho(B^w) = \rho(B) = \rho(\mathcal{B}(S')) = 1$, a contradiction.

Let U be an arbitrary absolute irreducible among the $P_{i,j}$ in the longest factorization of B^w. Then $\mathrm{Supp}(U) = \{m + g, -n + h\}$ for some $m, n \in \mathbb{N}$ and $g, h \in H$ by Lemma 3.7. If U also appears in the shortest factorization (this would occur, for instance, if $-n + h$ were the sole element of $\mathrm{Supp}(B)$ with a negative integer component), then $\rho(B^w) = \rho(B) = \rho(\mathcal{B}(S')) = 1$ by Proposition 3.3, a contradiction. Hence we may choose distinct absolute irreducibles V, W in the shortest factorization of B^w such that $V \neq U$, $W \neq U$, $\mathrm{Supp}(V) \supseteq \{m_i + g_i\}$ and $\mathrm{Supp}(W) \supseteq \{-n_j + h_j\}$. Set $T = \mathrm{Supp}(V) \cup \mathrm{Supp}(W)$, which has exactly two "negative" and two "positive" elements by the characterization of absolute irreducibles

(Lemma 3.7). Hence T is a nice subset and $\mathcal{B}(T)$ is strongly taut and half-factorial by assumption.

Taking $y = \max\{m_i \exp(H), n_j \exp(H)\}$, we guarantee that U divides $(VW)^y$ in $\mathcal{B}(T)$. By Lemma 3.7, we may choose an $z \in \mathbb{N}$ such that the zero-sum sequence $((VW)^y/U)^z$ factors into absolute irreducibles. Appending U^z to this factorization, we obtain a factorization F of $(VW)^{yz}$ into absolute irreducibles in which U appears. By the half-factoriality of $\mathcal{B}(T)$, this factorization must have length $2yz$.

Consider B^{wyz}. By Propositions 3.3 and 3.2, $L(B^{wyz}) = yzL(B^w)$ and $l(B^{wyz}) = yzl(B^w)$. As a longest factorization of B^{wyz}, take the longest factorization of B^w from equation 4.1 and repeat it yz times. This factorization consists entirely of absolute sequences and contains U. Analogously, the shortest factorization of B^w in equation 4.1 repeated yz times yields a shortest factorization of B^{wyz}. This factorization will have an occurrence of $(VW)^{yz}$, which we may substitute with the factorization F of $(VW)^{yz}$. This substitution does not change the length of the factorization of B^{wyz}, so it yields an alternate shortest factorization of B^{wyz} which contains U. So B^{wyz} has U in common between a longest and a shortest factorization, a contradiction by Proposition 3.3 since $\rho(B^{wyz}) = \rho(B) = \rho(\mathcal{B}(S')) > 1$. Since we have achieved contradiction in all cases, $\rho(\mathcal{B}(S')) = \rho(\mathcal{B}(S')) = 1$ and our block monoid is half-factorial. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. Let M be a Krull monoid, $\varphi\colon M \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group having torsion-free rank at most one, and $G_P \subseteq G$ the set of classes containing prime divisors. By Lemma 2.3, it is sufficient to prove the assertion for the block monoid $\mathcal{B}(G_P)$ instead of doing it for M, since properties such as strong tautness and half-factoriality transfer over. But the equivalence of these properties for $\mathcal{B}(G_P)$ is immediate from Theorem 4.4. □

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