# Breathe...

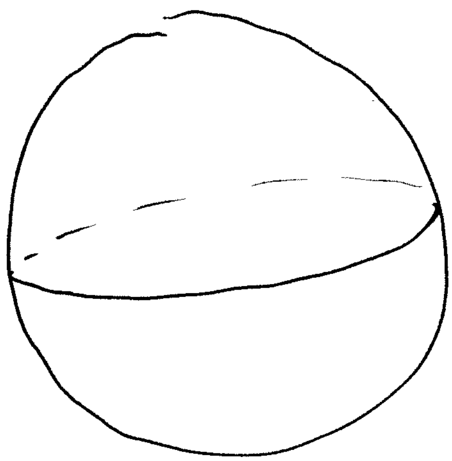
Math 171A
Fall 2009
Instructor: Shawn Rafalski

Differential Calculus
Exam 2!!

**Write your name on this exam right now.** Your work on this exam is to be your work alone. No calculators allowed. You have one hour to finish. Explain your answers clearly, and *show your work.* This exam has 9 pages, and the questions are worth a total of 100 points (not including bonus points). Only work on the bonus questions **after** you have tried to do all the regular questions. Don't forget to breathe regularly, and good luck!!

Begin working on the next page.

1. (15 points) A spherical balloon is being inflated by air in such a way that its radius is increasing at a rate of 2 cm/s. What is the rate of change of the volume of this balloon at the moment when the radius is 3 cm? (Note: the volume of the sphere of radius $r$ is $(4\pi r^3)/3$.



$$V = \frac{4\pi r^3}{3}$$

$$\frac{dr}{dt} = 2$$

Q: $\frac{dV}{dt} = ?$ when $r=3$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi \cdot 3^2 \cdot 2 = \boxed{72\pi \frac{cm^3}{s}}$$

2. (10 points each) Compute the derivatives of the following functions. Do not simplify your answers.

(a) $f(\theta) = \sec 2\theta$

$$f'(\theta) = 2 \sec 2\theta \tan 2\theta$$

(b) $g(x) = \dfrac{(x-1)(x-4)}{(x-2)(x-3)}$

$$g'(x) = \frac{[(x-4)+(x-1)](x-2)(x-3) - (x-1)(x-4)[(x-3)+(x-2)]}{[(x-2)(x-3)]^2}$$

(c) $p(t) = \sin^2\left(\cos\left(\sqrt{\sin \pi t}\right)\right)$

$$p'(t) = 2\sin\left(\cos\left(\sqrt{\sin \pi t}\right)\right) \cdot \cos\left(\cos\left(\sqrt{\sin \pi t}\right)\right) \cdot \left(-\sin\left(\sqrt{\sin \pi t}\right)\right) \cdot$$

$$\frac{1}{2}(\sin \pi t)^{-1/2} \cdot (\cos \pi t) \cdot \pi$$

3. The following limit represents the derivative of a function $f(x)$ at the point $x = a$.

$$\lim_{h \to 0} \frac{(1+h)^{17} - 1}{h}.$$

(a) (5 points) What is $f(x)$?

$f(x) = x^{17}$

(b) (5 points) What is $a$?

$a = 1$

(c) (5 points) What is the value of this limit?

$f'(x) = 17x^{16}$

$f'(1) = 17$

4. The equation $2x^3 + x^2y + xy^3 = -2$ defines $y$ as an implicit function of $x$.

(a) (10 points) Find $dy/dx$.

$$6x^2 + 2xy + x^2\frac{dy}{dx} + y^3 + x \cdot 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-6x^2 - 2xy - y^3}{x^2 + 3xy^2}$$

(b) (5 points) This equation also represents a curve in the plane. Calculate the slope of the tangent line to this curve at the point $(-1, 0)$.

Plug in $x = -1$, $y = 0$

$$\frac{dy}{dx} = -6$$

5. (10 points) Estimate the value of $\sqrt[4]{1.1}$ using a linear approximation. (If you are interested in knowing how close your approximation is, the actual value of $\sqrt[4]{1.1}$ is 1.024113689...)

$f(x) = x^{1/4}$ $\quad a = 1 \quad$ $f'(x) = \frac{1}{4} x^{-3/4}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = 1 + \frac{1}{4}(x-1)$

$L(1.1) = 1 + \frac{1}{4}(0.1) = \boxed{1.025}$

6. A particle moves along the $x$-axis so that its $x$ coordinate at time $t$ seconds is given by the function $x(t) = t^3 - 12t + 3$, where $t \geq 0$.

(a) (5 points) Calculate the velocity and acceleration functions for this motion.

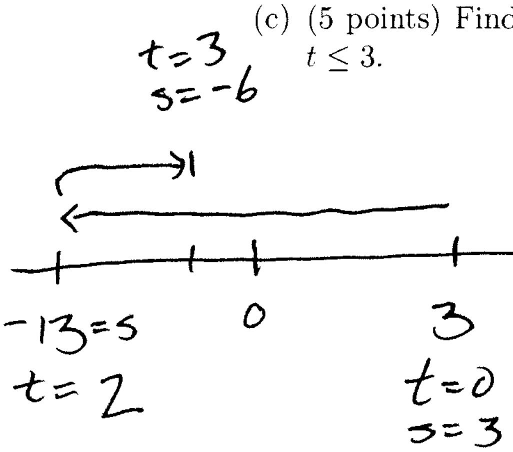
$v(t) = 3t^2 - 12 \qquad a(t) = 6t$

(b) (5 points) When is the particle moving forward? When is it moving backward?

$v(t) = 0$ when $t = \pm 2$ ($t = 2$ b/c $-2$ seconds doesn't make sense)

$\underbrace{v < 0 \text{ on } (0,2)}_{\text{back}}$ and $\underbrace{v > 0 \text{ on } (2,\infty)}_{\text{forward}}$

(c) (5 points) Find the distance the particle travels in the time interval $0 \leq t \leq 3$.




From $t=0$ to $t=2$ the particle moved back from $+3$ to $-13$ (16 units)

From $t=2$ to $t=3$, it moved from $-13$ to $-6$ (7 units)

So $16 + 7 = 23$ units

7. (**Bonus 5 points!**) Evaluate the limit

$$\lim_{x \to 0} \frac{\sin 6x}{\sin 4x}.$$

(Hint: You may wish to use the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$).

$$\lim_{x \to 0} \frac{\sin 6x}{\sin 4x} = \lim_{x \to 0} \frac{6}{4} \cdot \frac{\frac{\sin 6x}{6x}}{\frac{\sin 4x}{4x}}$$

$$= \frac{6}{4} \cdot \frac{\lim_{x \to 0} \frac{\sin 6x}{6x}}{\lim_{x \to 0} \frac{\sin 4x}{4x}} = \frac{6}{4} \cdot \frac{1}{1} = \boxed{\frac{3}{2}}$$