Math 171A
Fall 2009
Instructor: Shawn Rafalski

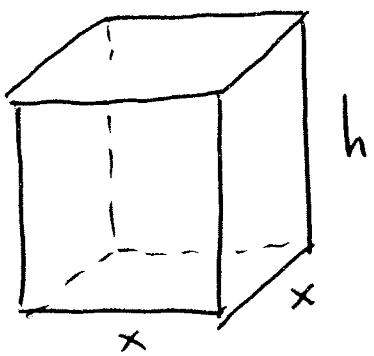
Differential Calculus
Exam 3!!

Write your name on this exam right now. Your work on this exam is to be your work alone. No calculators allowed. You have one hour to finish. Explain your answers clearly, and *show your work*. This exam has 10 pages, and the questions are worth a total of 100 points (not including bonus points). Only work on the bonus questions **after** you have tried to do all the regular questions. Don't forget to breathe regularly, and good luck!!

Solution

Begin working on the next page.

1. (15 points) Find the dimensions of the open-top cardboard box (this means that the box consists of only the base and the sides, not the lid) that has a square base, and whose volume is 4 ft^3, and that requires the minimum amount of cardboard for its construction.



$$V = x^2 h = 4 \longleftrightarrow h = \frac{4}{x^2}$$

$$\underbrace{S}_{\text{surface area}} = \underbrace{x^2}_{\text{base}} + \underbrace{4xh}_{\text{4 sides}}$$

$$S(x) = x^2 + 4x \cdot \frac{4}{x^2} = x^2 + \frac{16}{x}$$

$$S'(x) = 2x - \frac{16}{x^2} \qquad S' = 0 : \; 2x - \frac{16}{x^2} = 0$$

$$\downarrow$$

$$h = \frac{4}{(2)^2} = 1 \longleftarrow x = 2 \longleftarrow x^3 = 8$$

$$\boxed{x = 2 \quad h = 1}$$

2. (a) (15 points) Find the absolute extrema of the function $f(x) = x^3 - 3x + 1$ on the interval $[-2, 0]$.

$f' = 3x^2 - 3 \qquad f' = 0$ at $x = \pm 1$.

$x = 1$ is not in $[-2, 0]$

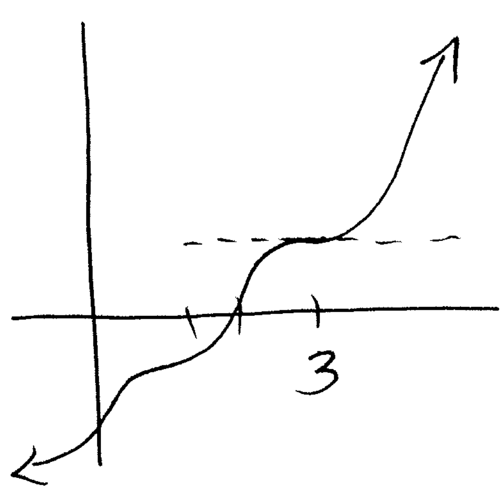
so disregard it.

$f(-2) = -1 \leftarrow$ abs min

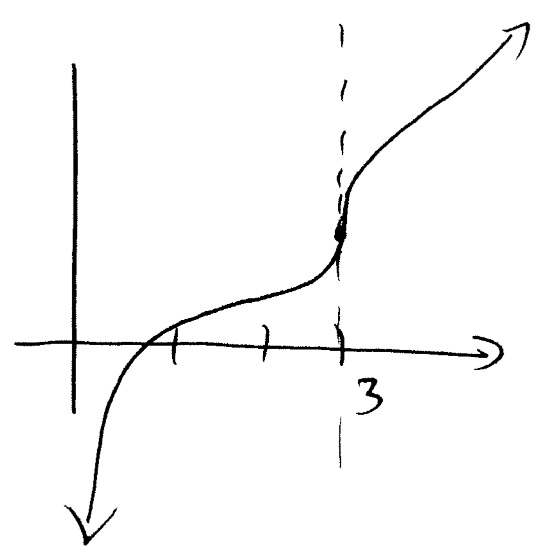
$f(-1) = 3 \leftarrow$ abs max

$f(0) = 1$

(b) (5 points) Draw the graph of a function that has $x = 3$ as a critical number, but which does not have a local extremum there.



or



* $\boxed{\text{on } [0,2]}$ *

3. (a) (13 points) Verify that the function $f(x) = x^2 + 5x - 2$ satisfies all the hypotheses of the Mean Value Theorem, and then find all the numbers c that satisfy the conclusion of the Mean Value Theorem.

f is continuous on $[0,2]$ and differentiable on $(0,2)$ b/c f is a polynomial

$$\frac{f(2)-f(0)}{2-0} = 7$$

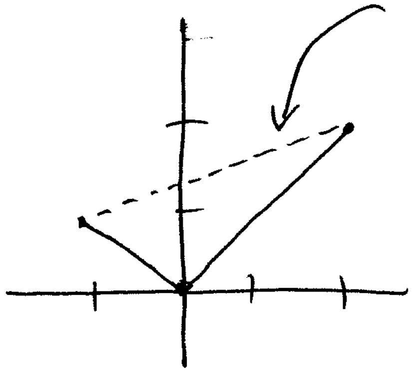
$$f'(c) = 2c+5$$

$$f'(c) = 7 \rightarrow 2c+5 = 7$$

$$\downarrow$$

$$\boxed{c = 1}$$

(b) (7 points) Consider the function $g(x) = |x|$ on the closed interval $[-1, 2]$. Show that there is no value of c between $x = -1$ and $x = 2$ that satisfies the conclusion of the Mean Value Theorem for g on this interval, and explain why this does not contradict the Mean Value Theorem.



$$\frac{g(2) - g(-1)}{2-(-1)} = \frac{1}{3}$$

There is no value between -1 & 2 such that $g'(c) = \frac{1}{3}$ b/c g is not differentiable at $x=0$.

4. (15 points) Find all the critical numbers of the function $f(x) = x^4 - 4x^3$ and determine whether they are local maxima, local minima, or neither.

$$f' = 4x^3 - 12x^2 = 4x^2(x-3)$$

$f' = 0$ at $x = 0, 3$

$f' < 0$		$f' < 0$		$f' > 0$
$f \searrow$		$f \searrow$		$f \nearrow$
	0		3	

0: not a local max or min

3: local min

5. (10 points) Find the limit

$$\lim_{x \to -\infty} \frac{x^3 - 2x^2 + 5}{7 - x^2}.$$

$$= \lim_{x \to -\infty} \frac{x^3 \left[1 - \frac{2}{x} + \frac{5}{x^3}\right]}{x^3 \left[\frac{7}{x^3} - \frac{1}{x}\right]}$$

$$= \lim_{x \to -\infty} \frac{1 - \frac{2}{x} + \frac{5}{x^3}}{\frac{7}{x^3} - \frac{1}{x}} \to \frac{1}{0}$$

This limit will be $\boxed{+\infty}$, b/c the leading terms $\frac{x^3}{-x^2}$ make a positive ratio as $x \to -\infty$.

6. Consider the function $f(x) = (x+1)^5 - 5x - 2$.

(a) (7 points) Find the intervals on which f is increasing or decreasing, and identify any local maxima or minima.

$$f' = 5(x+1)^4 - 5$$

$$f' = 0 \longrightarrow (x+1)^4 = 1 \rightarrow x = 0, -2$$

$f' > 0$ | $f' < 0$ | $f' > 0$

$f \nearrow$ | $f \searrow$ | $f \nearrow$

-2 0

$\uparrow$ local max $\uparrow$ local min

$f \nearrow$ on $(-\infty, -2)$ & $(0, \infty)$

$f \searrow$ on $(-2, 0)$

(b) (7 points) Find the intervals on which f is concave up or concave down, and identify any inflection points.

$f'' = 20(x+1)^3 \qquad f'' = 0 \rightarrow x = -1$

$f'' < 0 \quad | \quad f'' > 0$

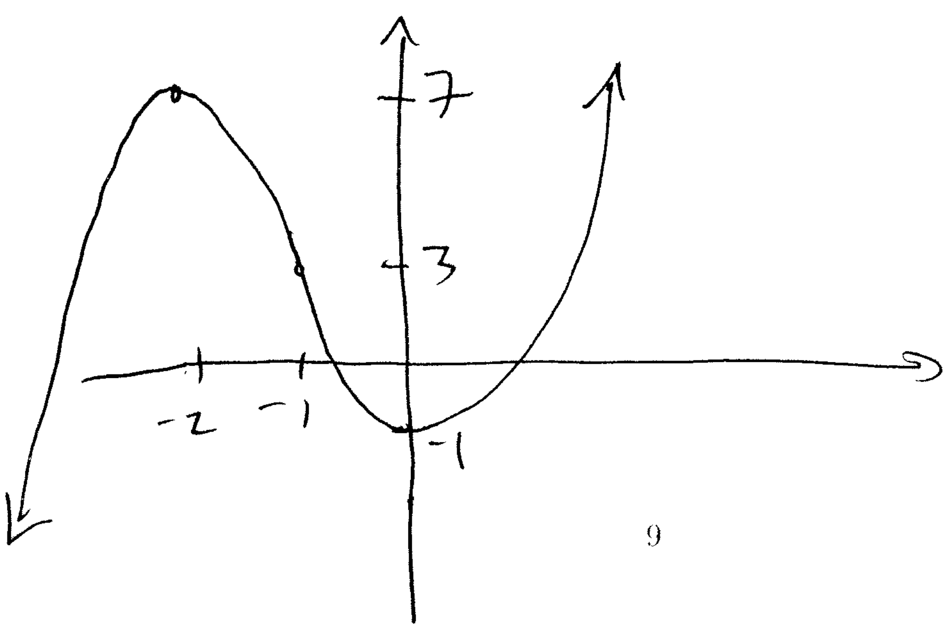
-1

$f \cup$ on $(-1, \infty)$

$f \cap$ on $(-\infty, -1)$

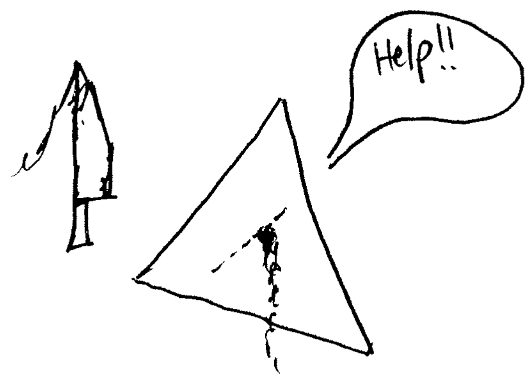
$x=-1$ inflection pt.

(c) (6 points) Sketch the graph of this function, using the information from parts (a) and (b).



7. (**Bonus 1 points!**) If a cube and a pyramid have the same volume, is it possible to cut the cube into a finite number of pieces and reassemble it into the pyramid?

Not necessarily. They have to have the same evil Dehn invariant.



8. (**Bonus 3 points!**) What are the two quantities of a 3-dimensional polyhedron that are used to compute its Evil Dehn Invariant?