AHHHHHHHH!!!!!!!!!!

Math 271A
Fall 2009
Instructor: Shawn Rafalski

Multivariable Calculus I
Exam 1

Write your name on this exam right now. Your work on this exam is to be your work alone. No calculators allowed. You have one hour to finish. Explain your answers clearly, and *show your work*. This exam has 7 pages, and the questions are worth a total of 100 points (not including bonus points). Only work on the bonus questions **after** you have tried to do all the regular questions. Don't forget to breathe regularly, and good luck!!

Here is some info you may (or may not) need for this exam.

$$\sin 0 = 0 = \cos\frac{\pi}{2}$$

$$\sin\frac{\pi}{2} = 1 = \cos 0$$

$$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

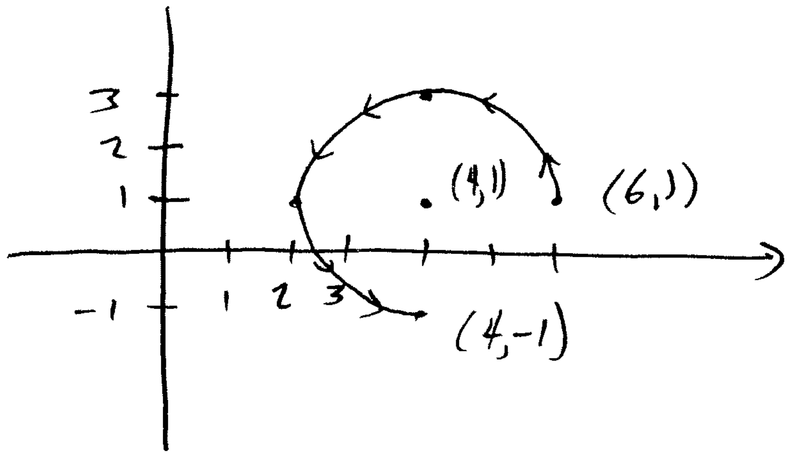
$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

Begin working on the next page.

1. Let C be the parametric curve in the plane described by $x = 4 + 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 3\pi/2$.

   (a) (10 points) Describe this curve, or provide a sketch.



   (b) (15 points) Calculate the slope of the tangent line to this curve at the point where $t = \pi/3$.

at $t = \pi/3$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} \rightsquigarrow \frac{2\cos \pi/3}{-2\sin \pi/3} = \frac{2\cdot 1/2}{-2\cdot \frac{\sqrt{3}}{2}}$$

$$\boxed{-\frac{1}{\sqrt{3}}}$$

(c) (5 points) Without actually doing the calculation, say in words the steps you would take to determine where this curve has vertical or horizontal tangent lines.

set $\frac{dy}{dt} = 0 \rightsquigarrow$ horizontal tangents

set $\frac{dx}{dt} = 0 \rightsquigarrow$ vertical tangents

(d) (10 points) Write down the equation you would need to solve in order to determine the value T for which the arc length of the portion of C from $t = 0$ to $t = T$ is 5 units. (**Bonus 2 points:** Solve this equation in order to find T).

$$5 = \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^T \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$

$$= \int_0^T 2\, dt = 2T.$$

So $T = 5/2$

2. Consider the 3-dimensional vectors $\mathbf{a} = \langle 1, 1, 2 \rangle$, $\mathbf{b} = \langle 0, 4, 1 \rangle$ and $\mathbf{c} = \langle -2, 1, 3 \rangle$ based at the origin.

(a) (10 points) Calculate the projection of $\mathbf{b}$ onto $\mathbf{a}$.

$$\mathrm{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\right)\vec{a} = \left(\frac{6}{\left(\sqrt{1+1+4}\right)^2}\right)\langle 1,1,2 \rangle$$

$$= \langle 1,1,2 \rangle .$$

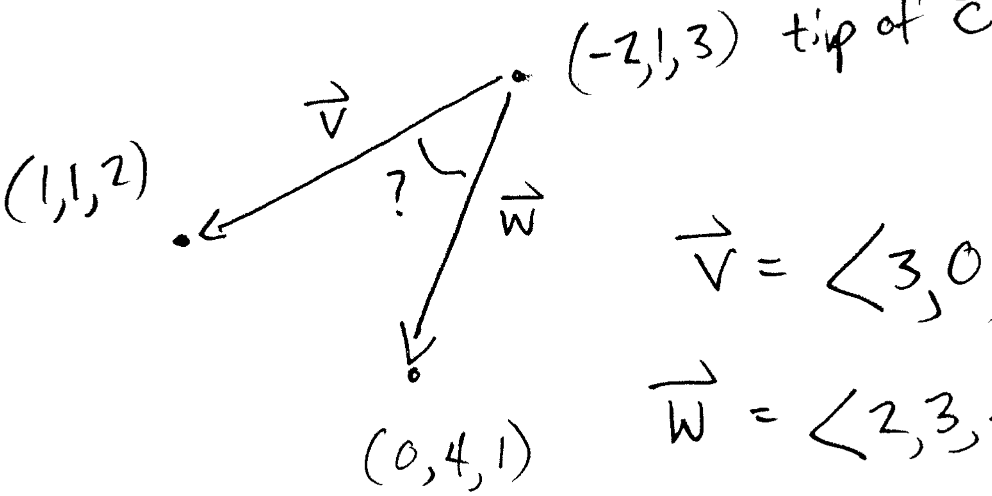
(b) (15 points) The vectors $\mathbf{b}$ and $\mathbf{c}$ form a triangle with vertices at the origin and the points (0,4,1) and (-2,1,3). Compute the area of this triangle.

$$\text{Area of triangle} = \frac{|\vec{b} \times \vec{c}|}{2}$$

$$\vec{b} \times \vec{c} : \left.\begin{array}{cccccc} i & j & k & i & j & k \\ 0 & 4 & 1 & 0 & 4 & 1 \\ -2 & 1 & 3 & -2 & 1 & 3 \end{array}\right\} \to \langle 11, -2, 8 \rangle$$

$$\frac{|\vec{b} \times \vec{c}|}{2} = \frac{\sqrt{121 + 4 + 64}}{2} = \frac{\sqrt{189}}{2}$$

(c) (15 points) Let T be the triangle whose vertices are the tips of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Determine whether or not the angle of T at the tip of $\mathbf{c}$ is a right angle.



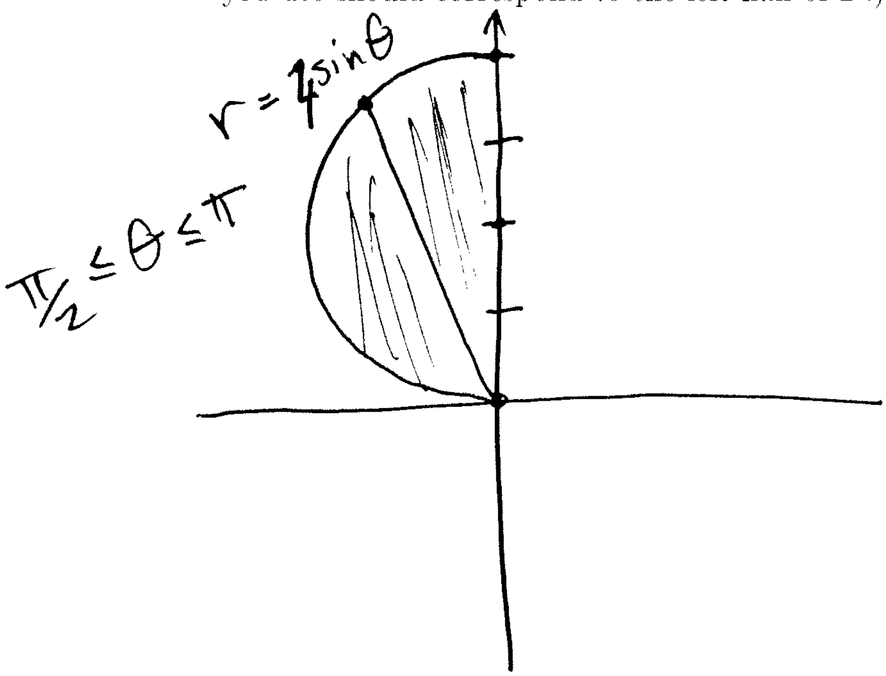
$$\vec{v} = \langle 3, 0, -1 \rangle$$

$$\vec{w} = \langle 2, 3, -2 \rangle$$

$$\vec{v} \cdot \vec{w} = 6 + 0 + 2 = 8 \neq 0.$$

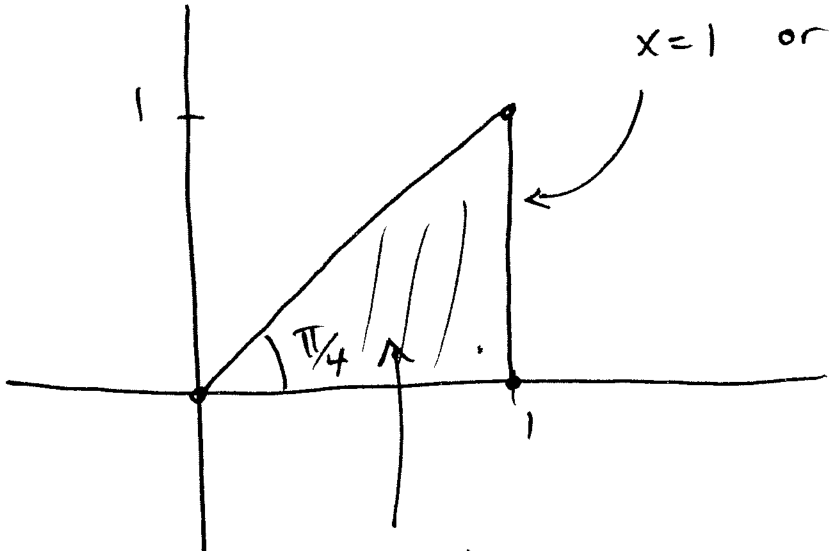
So the angle at $(-2,1,3)$ is <u>not</u> right.

3. (20 points) Let D be the region bounded by the circle of radius 2 centered at the point with Cartesian coordinates (0,2) in the plane. Set up, *but do not evaluate*, a polar coordinates integral that represents the area of the **left half** of D. (*Caution:* In order to get full credit for your answer, the limits of integration you use should correspond to the left half of D.)



$$\int_{\pi/2}^{\pi} \frac{(4\sin\theta)^2}{2}\, d\theta$$

4. (**Bonus 5 points**) Set up, but do not evaluate, a polar coordinates integral that represents the area of the triangle with Cartesian coordinates (0,0), (1,0) and (1,1).



$x = 1$ or $r\cos\theta = 1$

$$r = \frac{1}{\cos\theta}$$

$$r = \frac{1}{\cos\theta} \qquad 0 \le \theta \le \pi/4$$

$$\int_0^{\pi/4} \frac{1}{2}\left(\frac{1}{\cos\theta}\right)^2 d\theta$$