Don't Panic!

Math 271A
Fall 2009
Instructor: Shawn Rafalski

Multivariable Calculus I
Exam 2!!

Write your name on this exam right now. Your work on this exam is to be your work alone. No calculators allowed. You have one hour to finish. Explain your answers clearly, and *show your work*. This exam has 9 pages, and the questions are worth a total of 100 points (not including bonus points). Only work on the bonus questions **after** you have tried to do all the regular questions. Don't forget to breathe regularly, and good luck!!

Begin working on the next page.

1. (a) (15 points) Calculate an equation for the plane that contains the points $P(2, 1, 1)$, $Q(-1, -1, 10)$ and $R(1, 3, -4)$.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -3, -2, 9 \rangle \times \langle -1, 2, -5 \rangle = \langle -8, -24, -8 \rangle$$

$$0 = \langle -8, -24, -8 \rangle \cdot \langle x-2, y-1, z-1 \rangle$$

$$0 = -8(x-2) - 24(y-1) - 8(z-1)$$

or $$0 = (x-2) + 3(y-1) + (z-1)$$

(b) (10 points) Find parametric equations for the line through Q that is perpendicular to the plane in (a).

ℓ has direction vecton $\vec{n} = \langle -8, -24, -8 \rangle$

so

$$x = -1 - 8t$$

$$y = -1 - 24t$$

$$z = 10 - 8t$$

(there are other correct answers)

(c) (10 points) Does the line from part (b) intersect the xy-plane? If not, why? If so, what are the coordinates of the intersection point?

xy-plane $\longleftrightarrow$ $z = 0$

$\downarrow$

$t = 5/4$ $\leftarrow$ $\boxed{\text{yes!}}$ $\leftarrow$ $10 - 8t = 0$

$x = -1 - 8 \cdot \frac{5}{4} = -11$

$y = -1 - 24 \cdot \frac{5}{4} = -31$

$z = 0$

$(-11, -31, 0)$ intersection point

2. Consider the vector function $\mathbf{r}(t) = \langle 1+5t, -t, 7+2t \rangle$, $t \in \mathbb{R}$.

(a) (8 points) Describe the curve determined by $\mathbf{r}(t)$, and say what its curvature is and why.

$\vec{r}(t)$ is a line, so its curvature is zero.

(b) (12 points) Reparametrize this curve with respect to arc length (starting from $t = 0$).

$$s(t) = \int_0^t |\langle 5, -1, 2 \rangle| \, du$$

$$= \int_0^t \sqrt{30} \, du = \sqrt{30}\, t$$

so $t = \frac{s}{\sqrt{30}}$.

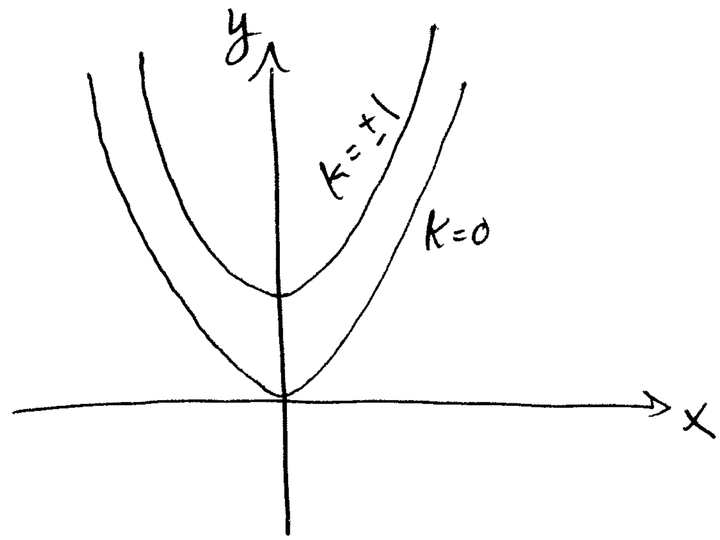
so $\vec{r}(s) = \left\langle 1 + 5\frac{s}{\sqrt{30}}, \frac{-s}{\sqrt{30}}, 7 + \frac{2s}{\sqrt{30}} \right\rangle$

is parametrized by length

3. Consider the surface in space determined by the equation $y = x^2 + z^2$

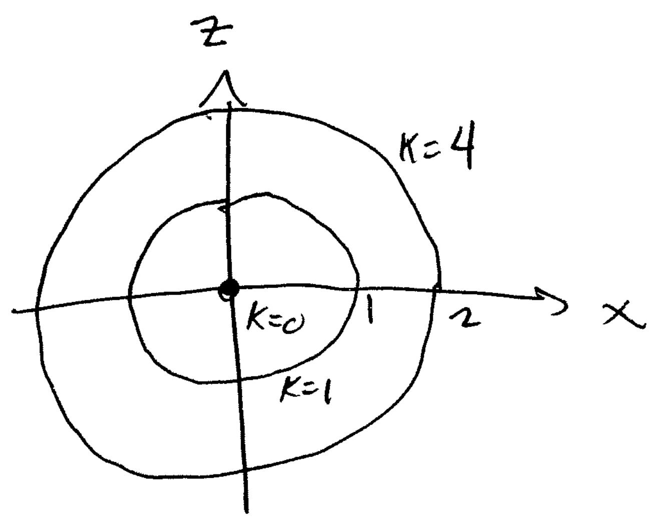
(a) (6 points) Draw the traces of this surface in the horizontal planes $z = k$ for $k = -1, 0$ and 1.

$y = x^2 + k^2$

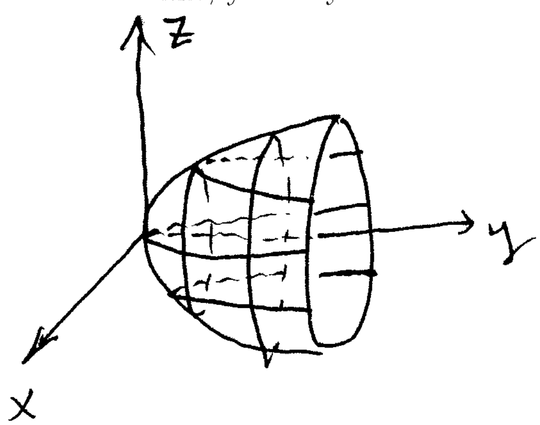


(b) (6 points) Draw the traces of this surface in the vertical planes $y = k$ for $k = 0, 1$, and 4.

$k = x^2 + z^2$



(c) (3 points) Describe in words what you think this object looks like. If you like, you may draw a sketch (but this is not required).



4. (10 points) Determine whether or not the planes given by $x - y + 2z = 4$ and $2x - 4y - z = 0$ are parallel. If they are not, then compute the angle of their intersection (leave your answer unsimplified).

They are not parallel b/c their normal vectors $\langle 1,-1,2\rangle$ & $\langle 2,-4,-1\rangle$ are not parallel.

The angle between the planes is the angle between the normal vectors:

$$2+4-2 = \langle 1,-1,2\rangle \cdot \langle 2,-4,-1\rangle = \sqrt{6}\sqrt{21}\cos\theta$$

$$\| \quad 4$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{6}\sqrt{21}}\right)$$

5. Consider the vector function $\mathbf{r}(t) = \langle 3t, 2\sin t, 2\cos t\rangle$ (t is any real number).

(a) (7 points) Calculate the unit tangent vector $\mathbf{T}(t)$ for this vector function.

$$\vec{r}\,'(t) = \langle 3, 2\cos t, -2\sin t\rangle \qquad |\vec{r}\,'(t)| = \sqrt{13}$$

$$\vec{T}(t) = \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\cos t, \frac{-2}{\sqrt{13}}\sin t \right\rangle$$

(b) (7 points) Calculate the curvature $\kappa(t)$ for this vector function.

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{\left|\left\langle 0, -\frac{2}{\sqrt{13}}\sin t, -\frac{2}{\sqrt{13}}\cos t \right\rangle\right|}{\sqrt{13}}$$

$$= \frac{\sqrt{4/13}}{\sqrt{13}} = \boxed{\frac{2}{13}}$$

(c) (6 points) Calculate the unit normal vector $\mathbf{N}(t)$ for this vector function.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\langle 0, -\frac{2}{\sqrt{13}}\sin t, -\frac{2}{\sqrt{13}}\cos t \rangle}{2/\sqrt{13}}$$

$$= \langle 0, -\sin t, -\cos t \rangle$$

6. (**Bonus 5 points**) Suppose that the curve C described by the vector function $\mathbf{r}(s)$ is parametrized by arc length. Show that if the curvature of C is zero, then C must be a straight line.

since $\vec{r}(s)$ is parametrized by length,

$$K(s) = \left| \frac{d\vec{T}}{ds} \right| .$$

Since $K(s) = 0$, then $\left| \frac{d\vec{T}}{ds} \right| = 0$ so

$\vec{T}$ = constant vector = $\langle a, b, c \rangle$

but $\vec{r}\,'(t) = \vec{T}(t) = \langle a, b, c \rangle$, so

b/c parametrized by length

$$\underbrace{\vec{r}(t) = \langle at, bt, ct \rangle + \text{constant vector}}_{\text{this is a line}}$$