In 3D!!

Solution!

Math 271A
Fall 2009
Instructor: Shawn Rafalski

Multivariable Calculus I
Exam 3: In 3D!!

Write your name on this exam right now. Your work on this exam is to be your work alone. No calculators allowed. You have one hour to finish. Explain your answers clearly, and *show your work.* This exam has 9 pages, and the questions are worth a total of 100 points (not including bonus points). Only work on the bonus questions **after** you have tried to do all the regular questions. Don't forget to breathe regularly, and good luck!!

Here is some info you may (or may not) need for this exam.

$$\sin 0 = 0 = \cos\frac{\pi}{2}$$

$$\sin\frac{\pi}{2} = 1 = \cos 0$$

$$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

Begin working on the next page.

1. (20 points) Let $f(x, y) = e^{xy}\sin(y^2)$. Calculate the partial derivatives $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial x^2}$.

$$f_y = xe^{xy}\sin(y^2) + e^{xy}\cos(y^2)\cdot 2y$$

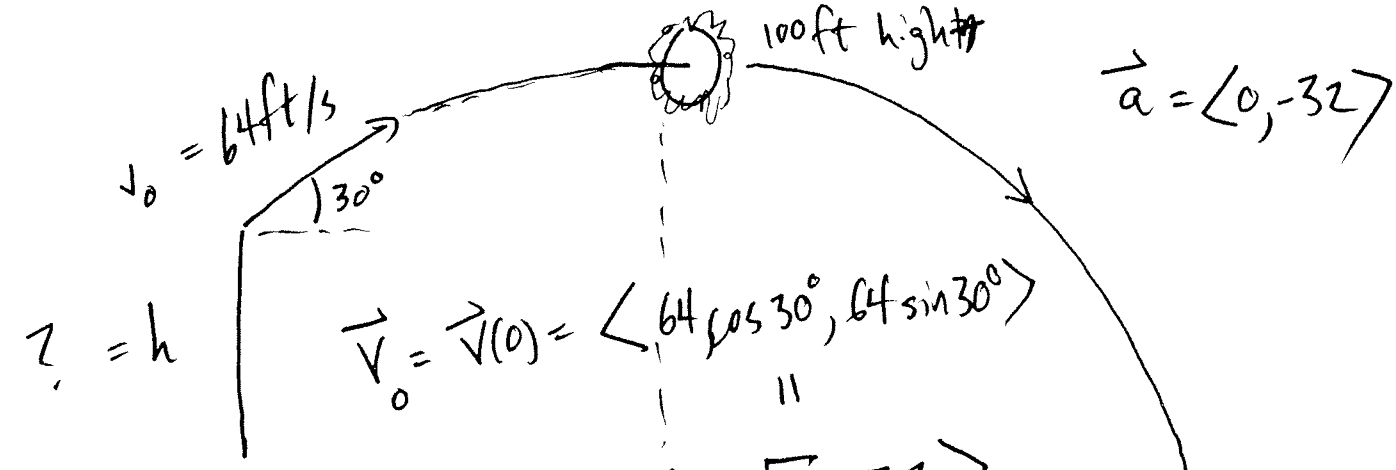
$$f_{yx} = e^{xy}\sin(y^2) + xye^{xy}\sin(y^2)$$
$$+ ye^{xy}\cos(y^2)\cdot 2y$$

$$f_x = ye^{xy}\sin(y^2)$$

$$f_{xx} = y^2e^{xy}\sin(y^2)$$

2. Suppose you are a human cannonball in the circus. You are going to be fired at 64 ft/s out of a cannon at an angle of 30°. At the highest point of your flight path, you plan to fly through a flaming hoop. The flaming hoop is 100 ft off the ground.

   (a) (20 points) Calculate the height from which you must be launched, in order to insure that the highest point on your path is the flaming hoop.



$\vec{v}_0 = \vec{v}(0) = \langle 64\cos 30°, 64 \sin 30° \rangle = \langle 32\sqrt{3}, 32 \rangle$

$\vec{r}_0 = \vec{r}(0) = \langle 0, h \rangle$

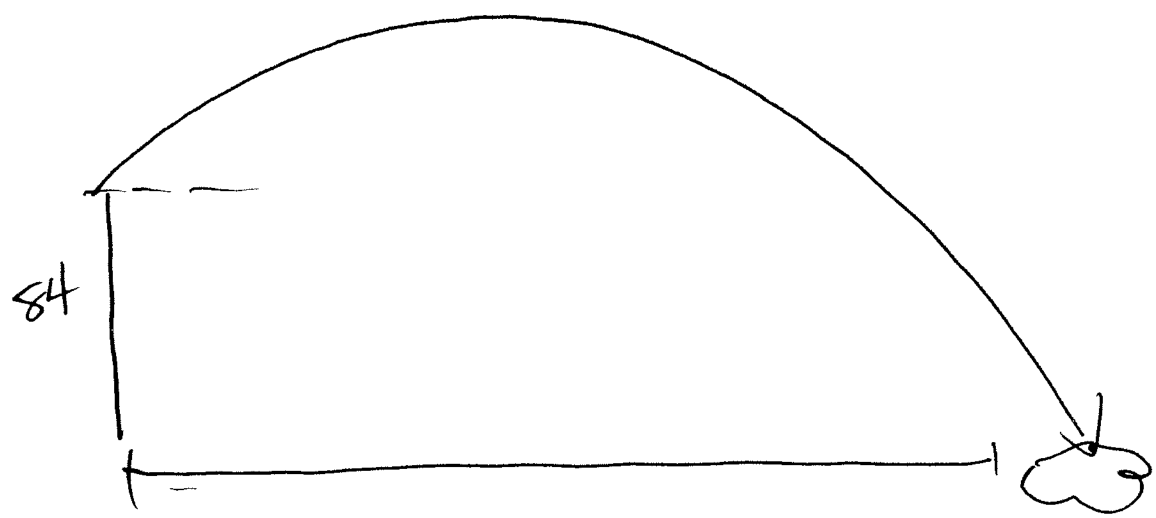
$\vec{v}(t) = \langle c_1, -32t + c_2 \rangle = \langle 32\sqrt{3}, -32t + 32 \rangle$

$\vec{r} = \langle 32\sqrt{3}\, t, -16t^2 + 32t + h \rangle$

The y-coordinate of $\vec{v}$ is zero at the highest point: $-32t + 32 = 0 \rightarrow t = 1$.

$\vec{r}(1) = \langle 32\sqrt{3}, \underbrace{-16 + 32 + h}_{= 100} \rangle \rightarrow \boxed{h = 84 \text{ ft}}$

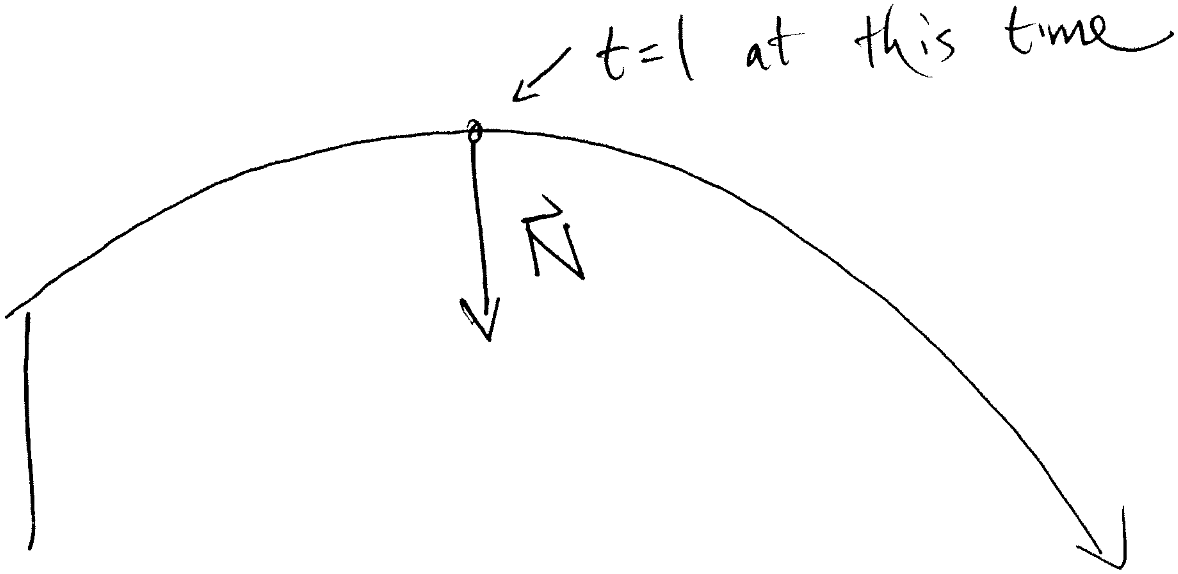
(b) (10 points) Write down in words the steps you would take in order to calculate where the landing cushion should be located on the ground, so that you can land safely. Do NOT do any calculations for this—you only need to say how you would proceed with this calculation. (In other words, your answer to this question should be the set of steps that I could follow, in order to know where to place the landing cushion.)



$\vec{r}(t) = \langle x(t), y(t) \rangle$

Solve $y(t) = 0$. Take the bigger value of t and plug it into $x(t)$.

(c) (5 points) If you know that the curvature at the highest point of your path is 1/96, then calculate the component of your acceleration in the direction of the unit normal vector **N** at that time.



$$\vec{a} = v'\vec{T} + \underbrace{v^2 \kappa}\,\vec{N}$$

$$\frac{v^2}{96} \quad \text{at } t = 1$$

$$\vec{v} = \langle 32\sqrt{3}, 0 \rangle$$

$$v(1) = |\langle 32\sqrt{3}, 0\rangle| = 32\sqrt{3}$$

$$v^2 = 32^2 \cdot 3$$

$$\frac{v^2}{96} = \frac{32 \cdot \cancel{32 \cdot 3}}{\cancel{96}} = \boxed{32}$$

3. (15 points) Suppose that $f = f(x, y, z, w)$, $x = x(q, r, t)$, $y = y(q, r, t)$, $z = z(q, r, t)$ and $w = w(q, r, t)$ are all differentiable functions. Write down the formula for the partial derivative $\partial f/\partial r$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial r}$$

4. (15 points) Use a tangent plane approximation to estimate the value of $e^{0.1}\sqrt{4.08}$. (Hint and Deal: You need to choose a function in order to answer this question. If you get completely stuck on this, then you can ask me for help and I will tell you what the function should be, but I will deduct 5 points in exchange for the hint.)

$f(x,y) = e^x\sqrt{y}$ $\qquad (a,b) = (0,4)$

$\uparrow$ close to $(0.1, 4.08)$

$f_x = e^x\sqrt{y}$

$f(0,4) = 2$

$f_y = \frac{1}{2}e^x \cdot \frac{1}{\sqrt{y}}$

$f_x(0,4) = 2 \qquad f_y(0,4) = \frac{1}{4}$

$z = 2 + 2(x-0) + \frac{1}{4}(y-4)$

$\downarrow$

$2 + 2(0.1-0) + \frac{1}{4}(4.08-4) = \boxed{2.22}$

5. (15 points) Let $T(x, y)$ be the temperature in °C at the point (x, y) in the plane. An insect is crawling around the xy-plane along the path determined by the vector function $\mathbf{r}(t) = \langle t^2, \sin \pi t \rangle$, where t is the time in seconds. If you know that $T_x(4, 0) = 3$ and $T_y(4, 0) = -2$, then what is the rate of change of the insect's temperature when it is at the point $(4, 0)$?

$$T(t) = T(x(t), y(t)) = T(t^2, \sin \pi t)$$

$$\frac{dT}{dt} = T_x \frac{dx}{dt} + T_y \frac{dy}{dt}$$

$$= T_x \cdot 2t + T_y \cos(\pi t) \cdot \pi$$

$$\underset{\substack{\uparrow \\ \text{at} \\ t=2}}{=} T_x(4,0) \cdot 2 \cdot 2 + T_y(4,0) \cdot \pi$$

$$= 3 \cdot 4 - 2\pi = \boxed{12 - 2\pi \ {}^\circ C/s}$$

$$F(x,y,z) = xyz - z^3 - \cos x = 0$$

6. (**Bonus 3 points**) Calculate the equation of the tangent plane to the surface $xyz = z^3 + \cos x$ at the point $(\pi, 0, 1)$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(yz + \sin x)}{xy - 3z^2}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(xz)}{xy - 3z^2}$$

at $(\pi, 0, 1)$ $\quad \frac{\partial z}{\partial x} = 0$

$$\frac{\partial z}{\partial y} = \frac{-\pi}{-3} = \pi/3$$

so

$$z = 1 + \pi/3 (y - 0) = 1 + \frac{\pi}{3} y$$