Hw 7 Solns

15.6 #24

$$f(x,y,z) = \frac{x+y}{z} \qquad (1,1,-1)$$

The max rate of change occurs in the direction

$$\nabla f(1,1,-1) = \left\langle \frac{1}{z}, \frac{1}{z}, -\frac{(x+y)}{z^2} \right\rangle \Bigg|_{(1,1,-1)}$$

$$= \langle -1, -1, -2 \rangle$$

and the rate of change is $|\langle -1,-1,-2\rangle|$

$$= \sqrt{6}$$

15.6 #41 a

$$x^2 - 2y^2 + z^2 + yz = 2 \quad (\text{surface } S)$$

tangent plane to S at $(2,1,-1)$:

$$F(x,y,z) = x^2 - 2y^2 + z^2 + yz$$

$$\nabla F = \langle 2x, -4y+z, 2z+y \rangle$$

$$\nabla F(2,1,-1) = \langle 4, -5, -1 \rangle$$

tangent plane: $0 = \langle 4,-5,-1\rangle \cdot \langle x-2, y-1, z+1\rangle$

$$\Downarrow$$

$$\boxed{0 = 4(x-2) - 5(y-1) - (z+1)}$$

15.6
~~#55~~

Let (a,b,c) be a point on the cone,

so $\boxed{a^2+b^2=c^2.}$ $\circledast$

$F(x,y,z) = z^2-x^2-y^2 = 0$ defines a surface S

$\nabla F(x,y,z) = \langle -2x, -2y, 2z \rangle$

$\downarrow$

$\nabla F(a,b,c) = \langle -2a, -2b, 2c \rangle$ is $\perp$ to the tangent plane at (a,b,c), so

$0 = \nabla F(a,b,c) \cdot \langle x-a, y-b, z-c \rangle$

$\downarrow$

$0 = -2a(x-a) - 2b(y-b) + 2c(z-c)$

$\downarrow$

$0 = -2ax + 2a^2 - 2by + 2b^2 + 2cz - 2c^2$

$$\downarrow$$

$$0 = -2ax - 2by + 2cz + \underbrace{2(a^2+b^2-c^2)}_{= 0 \text{ by } (*)}$$

$$\downarrow$$

$$0 = ax + by + cz \qquad \left(\begin{array}{c}\text{tangent} \\ \text{plane eq'n}\end{array}\right)$$

Now observe that this equation has $x=0$, $y=0$, $z=0$ as a solution, so the plane passes through the origin.

15.7 #12

$f = xy + 1/x + 1/y$

Find/Classify the critical points.

$\nabla f = \langle y - 1/x^2, x - 1/y^2 \rangle$

$\nabla f = \vec{0}$: $\left.\begin{array}{l} y = 1/x^2 \\ x = 1/y^2 \end{array}\right\} \to \left.\begin{array}{l} y = 1/x^2 \\ x = \frac{1}{(1/x^2)^2} = x^4 \end{array}\right\} \to$

$\left.\begin{array}{l} y = 1/x^2 \\ x - x^4 = 0 \end{array}\right\} \to \left.\begin{array}{l} y = 1/x^2 \\ x(1-x^3) = 0 \end{array}\right\} \to \left.\begin{array}{l} y = 1/x^2 \\ x = 0 \text{ or } x = 1 \end{array}\right\} \to$

not in domain of f

$\left.\begin{array}{l} y = 1/1^2 = 1 \\ x = 1 \end{array}\right.$ so $(1,1)$ is the only critical point

$f_{xx} = 2/x^3$, $f_{yy} = 2/y^3$, $f_{xy} = 1$

So $D = \frac{2 \cdot 2}{x^3 y^3} - 1^2$.

$D(1,1) = 4 - 1 = 3 > 0$

$f(1,1) = 2 > 0$

So $(1,1)$ is a local min.

15.7 #32 $f = 4x + 6y - x^2 - y^2$

Find absolute extrema on

$$D = \begin{Bmatrix} 0 \le x \le 4 \\ 0 \le y \le 5 \end{Bmatrix}$$

$\nabla f = \langle 4 - 2x, 6 - 2y \rangle$

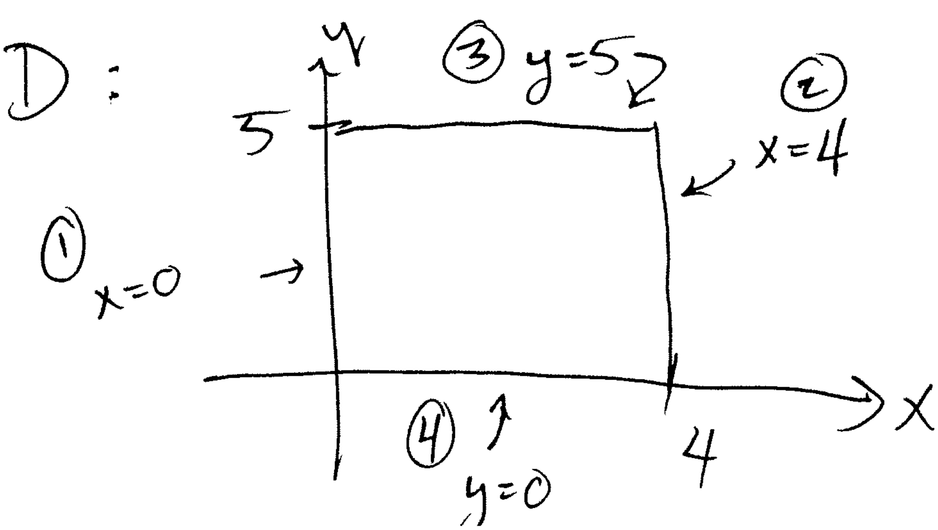
$\nabla f = \vec{0} \longrightarrow x = 2, y = 3$

so $\boxed{(2,3)}$ is the only critical point.

And $(2,3)$ is in D.

$f(2,3) = 4 \cdot 2 + 6 \cdot 3 - 4 - 9 = 13$

Now look at D:



① $x=0$: $f(0,y) = 6y - y^2 \quad 0 \le y \le 5$

$f'(0,y) = 6 - 2y = 0$ at $y = 3$

So check
$$\boxed{\begin{aligned} f(0,0) &= 0 \\ f(0,3) &= 9 \\ f(0,5) &= 5 \end{aligned}}$$

② $x=4$: $f(4,y) = 6y - y^2 \quad 0 \le y \le 5$

Same as ①:
$$\boxed{\begin{aligned} f(4,0) &= 0 \\ f(4,3) &= 9 \\ f(4,5) &= 5 \end{aligned}}$$

③ $\underline{y=5}$: $f(x,5) = 4x - x^2 + 5$

$0 \le x \le 4$

$f'(x,5) = 4 - 2x = 0$ at $x = 2$

so check

$$\boxed{\begin{aligned} f(0,5) &= 5 \\ f(2,5) &= 9 \\ f(4,5) &= 5 \end{aligned}}$$

④ $\underline{y=0}$: $f(x,0) = 4x - x^2$ $\quad 0 \le x \le 4$

Same as ③:

$$\boxed{\begin{aligned} f(0,0) &= 0 \\ f(2,0) &= 4 \\ f(4,0) &= 0 \end{aligned}}$$

Comparing all outputs ~~we see~~

$$\boxed{\begin{gathered} f(2,3) = 13 \\ \text{abs max} \end{gathered}} \qquad \boxed{\begin{gathered} f(0,0) = 0 = f(4,0) \\ \text{abs min} \end{gathered}}$$