Math 172A
Spring 2010
Instructor: Shawn Rafalski

Integral Calculus
Exam 1: "Best. Exam. Ever."

Write your name on this exam right now. Your work on this exam is to be your work alone. No calculators allowed. You have 75 minutes to finish. Explain your answers clearly, and *show your work.* This exam has 11 pages, and the questions are worth a total of 100 points (not including bonus points). Only work on the bonus questions **after** you have tried to do all the regular questions. Don't forget to breathe regularly, and good luck!!

Solution

Begin working on the next page.

1. (5 points each) Evaluate the following integrals. You do not have to simplify your answers.

   (a) $\int_0^1 (2x + x^3 - 2)\, dx$

$$= \left(x^2 + \frac{x^4}{4} - 2x\right)\Big|_0^1 = -3/4$$

   (b) $\int \sin 10t\, dt$

$u = 10t$
$du = 10dt$

$$= \frac{-\cos 10t}{10} + C$$

   (c) $\int \frac{\cos(1/x)}{x^2}\, dx$

$u = 1/x$
$du = -1/x^2\, dx$

$$= -\sin(1/x) + C$$

(d) $\int_1^2 \sqrt{2x-1}\,dx$

$u = 2x-1$

$du = 2\,dx$

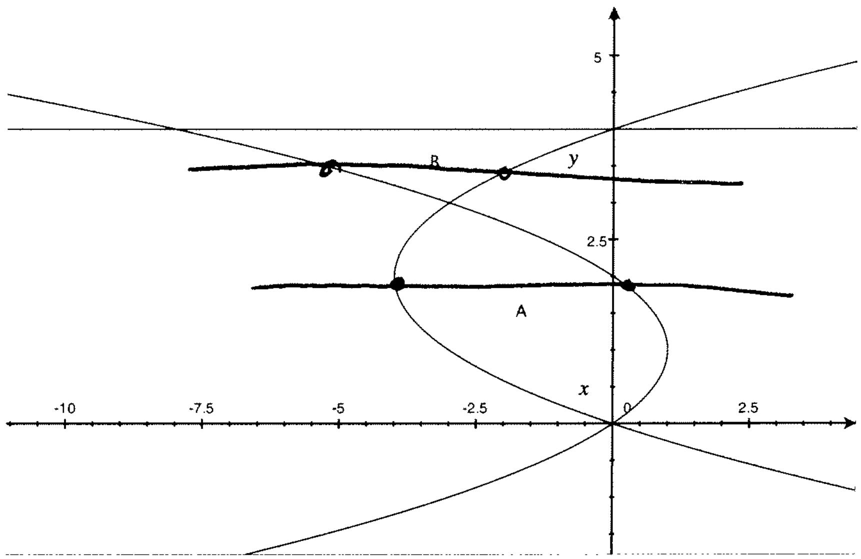
$$= \int_1^3 \tfrac{1}{2} u^{1/2}\,du = \tfrac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_1^3 = \tfrac{1}{3}\left(3^{3/2} - 1\right)$$

(e) $\int (\cos x)(\sin x)^7\,dx$

$u = \sin x$

$du = \cos x\,dx$

$$= \frac{(\sin x)^8}{8} + C$$



2. (15 points) The figure above shows the parabolas $x = y^2 - 4y$ (which opens to the right) and $x = 2y - y^2$, and the line $y = 4$. The parabolas intersect at the points $(-3, 3)$ and $(0, 0)$. Compute the area of the two regions A and B in the figure. You **do not** have to simplify your answer. You can use the next page if you need more workspace.

$$\text{Area}(A) + \text{Area}(B)$$

$$= \int_0^3 \left[(2y - y^2) - (y^2 - 4y)\right] dy + \int_3^4 \left[(y^2 - 4y) - (2y - y^2)\right] dy$$

$$= \int_0^3 (6y - 2y^2)\, dy + \int_3^4 (2y^2 - 6y)\, dy$$

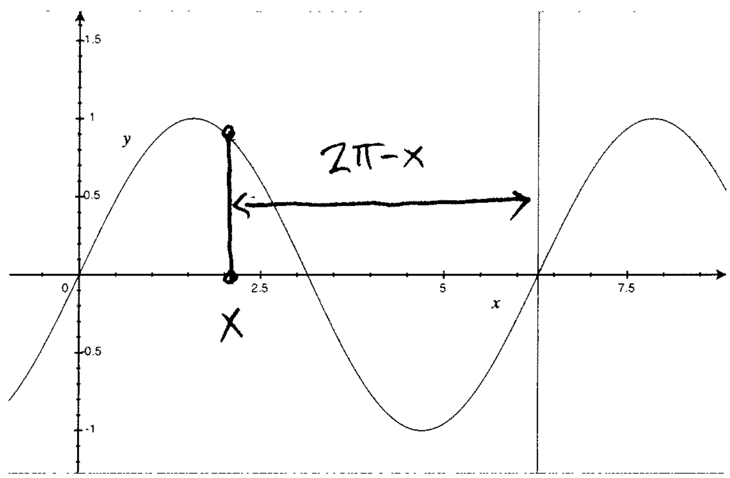
$$= \left(3y^2 - \frac{2}{3}y^3\right)\Big|_0^3 + \left(\frac{2}{3}y^3 - 3y^2\right)\Big|_3^4 = \cdots \longrightarrow$$

(Use this page for additional workspace.)

$$\longrightarrow \quad \cdots = \left(3\cdot 3^2 - \frac{2}{3}(3)^3 - 0\right)$$

$$+ \left(\frac{2}{3}(4)^3 - 3(4)^2 - \left(\frac{2}{3}(3)^3 - 3(3)^2\right)\right)$$

(you don't have to simplify this.)



3. The figure above shows the graph of $y = \sin x$ and the line $x = 2\pi$.

   (a) (10 points) Let D be the region bounded by the graph of $\sin x$ and the x–axis from $x = 0$ to $x = \pi$. Write down an integral, including the limits of integration, that represents the volume of the solid of revolution obtained by revolving D around the vertical line $x = 2\pi$. **Do not evaluate this integral.**

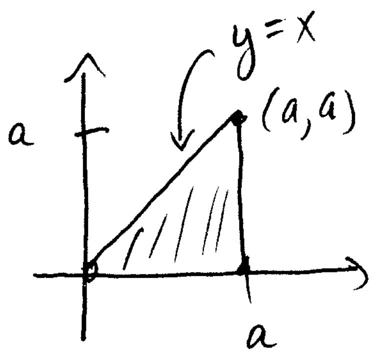
Cylindrical shells

$$\int_0^{\pi} 2\pi \cdot \underbrace{(2\pi - x)}_{\text{radius}} \cdot \underbrace{\sin x}_{\text{height}} \; \underbrace{dx}_{\text{"width"}}$$

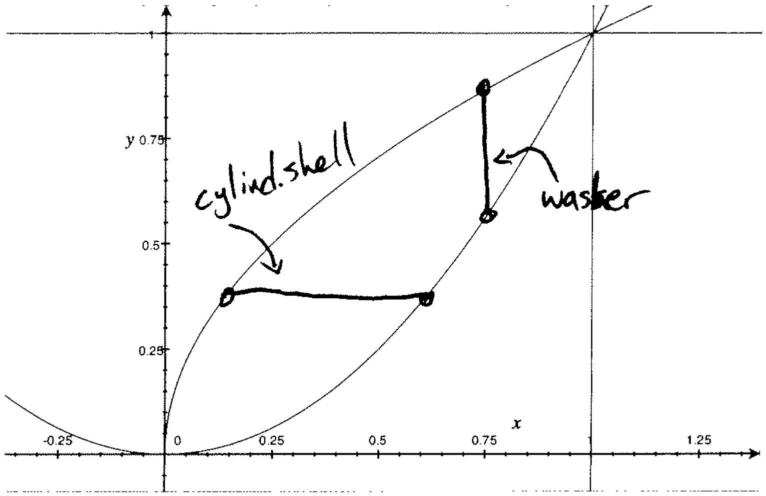
(b) (5 points) Compute the average value of $\sin x$ over the interval $[0, \pi]$.

$$\frac{1}{\pi - 0} \int_0^{\pi} \sin x \, dx = \frac{1}{\pi} \cdot 2 = \frac{2}{\pi}$$

4. (5 points) Let $a > 0$. Write down an integral that represents the area of the triangle in the plane with vertices $(0,0)$, $(a,0)$ and (a,a).



$$\rightsquigarrow \int_0^a x \, dx = \text{area}$$



5. The figure above shows the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, and the lines $x = 1$ and $y = 1$. Use this figure for the next three parts (for your convenience, it is reproduced on the next two pages).

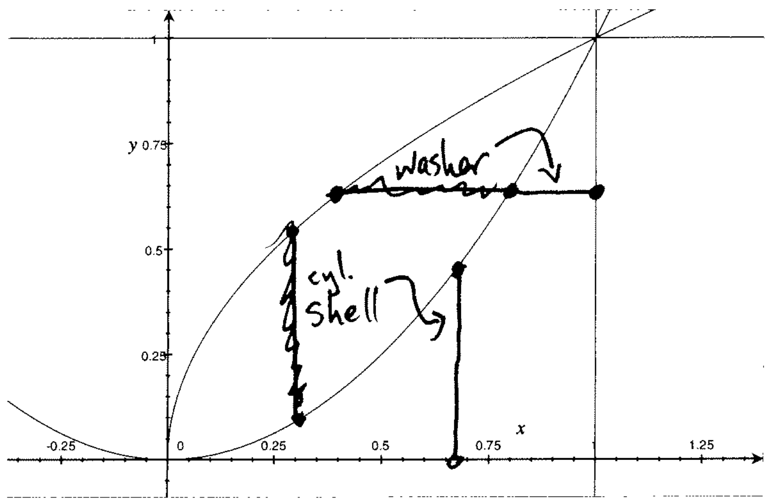
(a) (15 points) Let R be the region trapped between $y = x^2$ and $y = \sqrt{x}$. Write down an integral, including the limits of integration, that represents the volume of the solid obtained by revolving R around the x–axis. *You do not need to evaluate this integral.*

washers:

$$\int_0^1 \left[\pi(\sqrt{x})^2 - \pi(x^2)^2\right] dx$$

cylin. shells:

$$\int_0^1 2\pi \cdot (\sqrt{y} - y^2) \cdot y \; dy$$



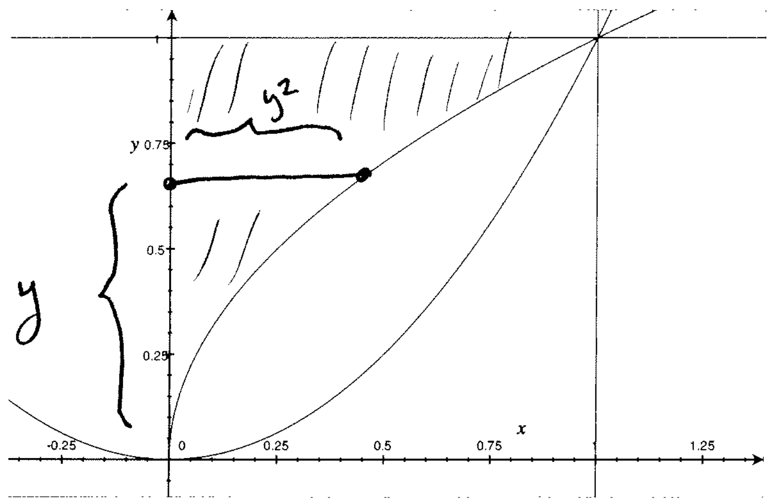
(b) (15 points) Let S be the region bounded by $y = x^2$, $x = 1$ and the x–axis. Write down an integral, including the limits of integration, that represents the volume of the solid obtained by revolving S around the y–axis. *You do not need to evaluate this integral.*

washers:

$$\int_0^1 \left[\pi(1)^2 - \pi(\sqrt{y})^2\right] dy$$

cyl. shells:

$$\int_0^1 2\pi \cdot x \cdot x^2 \, dx$$



(c) (10 points) The integral below represents the volume of the solid obtained by revolving some region from the figure around some line from the figure. Identify the region, and identify the line, and say whether the method being used to set up the integral is the method of "washers" or the method of "cylindrical shells."

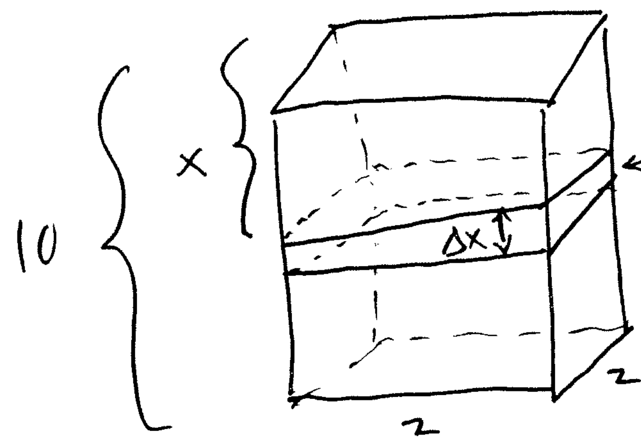
$$\int_0^1 2\pi y^3 \, dy$$

Region: bounded by $y^2 = x$, $y = 1$, $x = 0$

Line: x-axis, $(y = 0)$

Method: Cylin. Shells

6. **(5 Bonus points!)** A 10 ft tall rectangular box-shaped tank with a 2 ft $\times$ 2 ft square base is filled with smelly, vile, disgusting, industrial waste-style sludge that weighs 3 $\ell bs/\text{ft}^3$. Compute the work done in pumping all of the sludge out of the top of the tank.



← slab of height Δx ft weighs

$$\approx (2 \cdot 2 \cdot \Delta x)\,\text{ft}^3 \cdot (3)\,\frac{\text{lbs}}{\text{ft}^3} = 12\,\Delta x \text{ lbs}$$

moving this x ft takes $\approx 12\Delta x \cdot x$ ft·lbs

and x goes from 0 to 10

so $$\int_0^{10} 12x\,dx = 6x^2\Big|_0^{10} = \text{600 ft·lbs of work}$$