What the Flux!!?!

Math 272A
Spring 2010
Instructor: Shawn Rafalski

Multivariable Calculus II
Exam 3

Write your name on this exam right now. I am reminding you now that the Honor Code applies to your work on this exam. This means that your work on this exam is to be your work alone. Sign the statement below indicating that you have neither given nor received any aid on this exam (apart from the use of your book, notes and course materials). **You will not receive credit without signing the statement.** Explain your answers clearly, and *show your work*. This exam has 10 pages, and the questions are worth a total of 100 points. Good luck!!

SOLUTIONS

"I, Shawn T. Rafalski, have neither given nor received aid of any kind on this exam (apart from the use of my book, notes and course materials)."

Begin working on the next page.

(1) (5 points) Consider the vector field

$$\mathbf{G}(x, y, z) = \langle x\cos z, y, z^3\rangle.$$

Show either that there is no vector field $\mathbf{F}$ such that $\mathbf{G} = \text{curl } \mathbf{F}$, or else find such a vector field.

$\text{div}\, G = \cos z + 1 + 3z^2 \neq 0$

so $G \neq \text{curl}\, \vec{F}$, b/c if

$G = \text{curl}\, \vec{F}$ then

$\text{div}\, G = \text{div}(\text{curl}\, \vec{F}) = 0$

(this is a theorem we learned)

(2) (8 points) Determine whether or not the vector field

$$\mathbf{F}(x, y, z) = \langle y\cos(xy), x\cos(xy), -\sin z\rangle$$

is conservative. If it is conservative, then find a potential function for **F**.

$$\begin{array}{cccccc} i & j & k & i & j & k \\ dx & dy & dz & dx & \underline{dy} & dz \\ y\cos xy & x\cos xy & -\sin z & y\cos xy & \cancel{x}\cos xy & -\sin z \end{array}$$

$$\text{curl}\, F = \langle 0-0, 0-0, -xy\sin xy + \cos xy - \cos xy + xy\sin xy\rangle$$

$$= \langle 0,0,0\rangle$$

pot:

$f_x = y\cos xy$

$f_y = x\cos xy$

$f_z = -\sin z \rightarrow f = \cos z + g(x,y) \rightarrow f_x = \frac{\partial g}{\partial x} = y\cos xy$

$\int dx$

$f = \cos z + \sin xy + h(y) \leftarrow g(x,y) = \sin xy + h(y)$

$f_y = x\cos xy + h'(y) = x\cos xy \rightarrow h(y) = \text{constant}$

So $\boxed{f = \cos z + \sin xy + \text{constant}}$

(3) (8 points) Let S be a closed surface (that is, a surface with no boundary), and let $\mathbf{c}$ be a constant vector field on 3–dimensional space. Show that

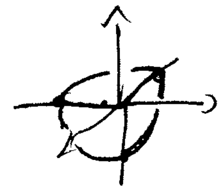
$$\iint_S \mathbf{c} \cdot d\mathbf{S} = 0.$$

$\operatorname{div} \vec{c} = 0$ so div thm says

integral $= \iiint_E 0 \, dV = 0$

(4) (12 points each) Use integration to compute the surface areas of the following surfaces. Show your work.

(a) The part of the sphere of radius 4 all of whose coordinates are less than or equal to zero.



$$S: \quad \vec{r}(\phi,\theta): \begin{cases} x = 4\cos\theta\sin\phi \\ y = 4\sin\theta\sin\phi \\ z = 4\cos\phi \end{cases} \qquad \begin{array}{l} \pi/2 \le \phi \le \pi \\ \pi \le \theta \le 3\pi/2 \end{array}$$

$$\vec{r}_\phi = \langle 4\cos\theta\cos\phi,\ 4\sin\theta\cos\phi,\ -4\sin\phi \rangle$$

$$\vec{r}_\theta = \langle -4\sin\theta\sin\phi,\ 4\cos\theta\sin\phi,\ 0 \rangle$$

$$\begin{array}{cccccc} i & j & k & i & j & k \\ 4\cos\theta\cos\phi & 4\sin\theta\cos\phi & -4\sin\phi & 4\cos\theta\cos\phi & 4\sin\theta\cos\phi & -4\sin\phi \\ -4\sin\theta\sin\phi & 4\cos\theta\sin\phi & 0 & -4\sin\theta\sin\phi & 4\cos\theta\sin\phi & 0 \end{array}$$

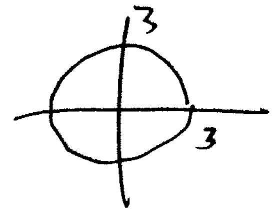
$$\vec{r}_\phi \times \vec{r}_\theta = \langle 16\cos\theta\sin^2\phi,\ 16\sin\theta\sin^2\phi,\ 16\sin\phi\cos\phi \rangle$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\left(16^2\sin^4\phi + 16^2\sin^2\phi\cos^2\phi\right)} = 16\sin\phi$$

$$\text{Area} = \int_{\pi}^{3\pi/2}\int_{\pi/2}^{\pi} 16\sin\phi \, d\phi \, d\theta = 16 \cdot \frac{\pi}{2} = \boxed{8\pi}$$

(b) The part of the plane $2x + 5y + z = 10$ that lies inside the radius 3 cylinder centered around the z-axis.

$z = 10 - 2x + 5y$

D: 

$0 \le r \le 3$
$0 \le \theta \le 2\pi$

$\vec{r}(x,y) = \langle x, y, 10 - 2x - 5y \rangle$

$|\vec{r}_x \times \vec{r}_y| = |\langle 2, 5, 1 \rangle| = \sqrt{4 + 25 + 1} = \sqrt{30}$

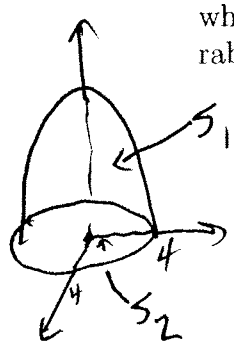
$$\text{Area} = \int_0^{2\pi} \int_0^3 \sqrt{30}\, r\, dr\, d\theta$$

$$= 2\pi \cdot \sqrt{30} \cdot \frac{9}{2} = \boxed{9\pi\sqrt{30}}$$

(5) (10 points) Evaluate the surface integral

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \langle yz^3, xz^2 + \sin(y^2), -x^2z/2\rangle$, and S is the part of the paraboloid $z = 16 - x^2 - y^2$ with non-negative z-coordinate.



$$\text{curl}\,\vec{F} = \langle -2xz,\ 3yz^2 + zx,\ z^2 - z^3 \rangle$$

$D = S_2$

$$\iint_{S_2} \text{curl}\,\vec{F} = \boxed{0} = \iint_{S_2} \text{curl}\,\vec{F} \cdot d\vec{S}$$

b/c $z=0$ on S_2

$\|$ ← 1 way to do this

S_2: $\vec{r}(x,y) = \langle x, y, 16 - x^2 - y^2 \rangle$

$$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$$

$$\iint_D \left(-4x^2z + 6y^2z^2 + 2xyz + z^2 - z^3 \right) dA$$

$$= \int_0^{2\pi}\int_0^4 (\text{plug in } z = 16 - x^2 - y^2)\ r\,dr\,d\theta \quad \leftarrow \boxed{\text{Hard Integra}}$$

$C = \partial D$: $\vec{r}(t) = \langle 4\cos t, 4\sin t, 0 \rangle \quad 0 \le t \le 2\pi$

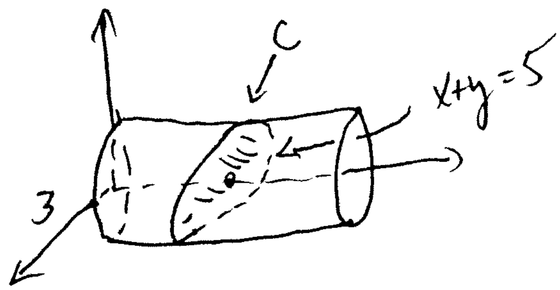
$$\int_0^{2\pi} \langle 0, \sin(16\sin^2 t), 0 \rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} 4\cos t \sin(16\sin^2 t)\, dt \ \ldots ?$$

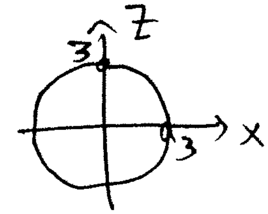
$\text{curl}\, F = \langle -1, x, 0 \rangle$

(6) Compute each of the following integrals (without the assistance of any device that is capable of computing integrals). Show your work.

(a) (15 points) $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xz, 3z, 2y \rangle$ and C is the curve of intersection of the plane $x + y = 5$ and the cylinder $x^2 + z^2 = 9$.



D :



$y(x,z) = 5 - x$

$$\int_{E} \vec{F} \cdot d\vec{r} = \iint_{D} \langle -1, x, 0 \rangle \cdot \langle 1, 1, 0 \rangle \, dA$$

$$= \int_0^{2\pi}\int_0^3 (x - 1) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^3 (r^2\cos\theta - r) \, dr \, d\theta$$

$$= \int_0^3 r^2 dr \int_0^{2\pi} \cos\theta \, d\theta - 2\pi \int_0^3 r \, dr = \boxed{-9\pi}$$

$C: \vec{r}(t) = \langle 3\cos t, 5 - 3\cos t, 3\sin t \rangle$

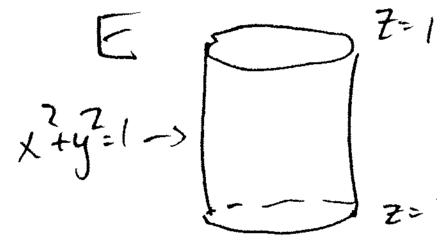
$$\int_0^{2\pi} \langle 9\sin t\cos t, 9\sin t, 10 - 6\cos t \rangle \cdot \langle -3\sin t, 3\sin t, 3\cos t \rangle \, d$$

$$= \cdots$$

(b) (15 points) $\iint_S \langle x^3, ze^x, 3zy^2\rangle \cdot d\mathbf{S}$, where S is the surface of the solid bounded by $x^2+y^2=1$ and the planes $z=-2$ and $z=1$.

Div Thm: $\text{Int} = \iiint_E 3x^2 + 3y^2 \, dV$

E



$= \int_0^{2\pi}\int_0^1\int_{-2}^1 3r^2 \cdot r \, dz\,dr\,d\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$-2 \le z \le 1$

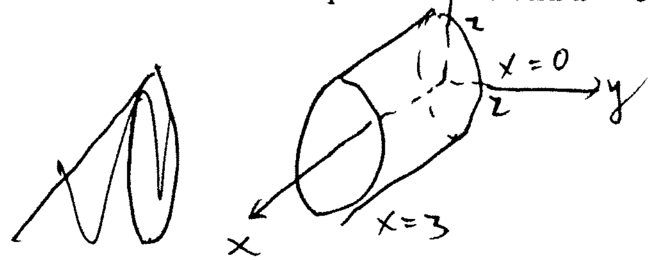
$= 2\pi \cdot 3 \cdot \frac{3}{4} = \boxed{\frac{9\pi}{2}}$

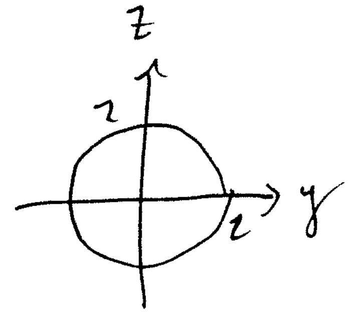
or

sum three surfaces

(c) (15 points) $\iint_S (x^2 + y^2 + z^2)\, dS$, where S is only the part of the cylinder $y^2 + z^2 = 4$ between the planes $x = 0$ and $x = 3$.

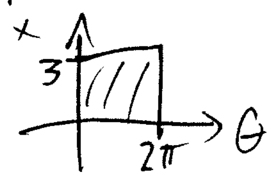
S :





$S : \vec{r}(\theta, x) = \langle x, 2\cos\theta, 2\sin\theta \rangle$

D :



$\vec{r}_\theta = \langle 0, -2\sin\theta, 2\cos\theta \rangle$

$\vec{r}_x = \langle 1, 0, 0 \rangle$ $\quad \vec{r}_\theta \times \vec{r}_x = \langle 0, 2\cos\theta, 2\sin\theta \rangle$

$|\vec{r}_\theta \times \vec{r}_x| = 2$

$$\text{Int} = \int_0^{2\pi}\int_0^3 2(x^2 + 4)\, dx\, d\theta$$

$$= 2 \cdot 2\pi \cdot \left(\frac{x^3}{3} + 4x \right)\Big|_0^3 = 2 \cdot 2\pi\,(9 + 12) = \boxed{84\pi}$$