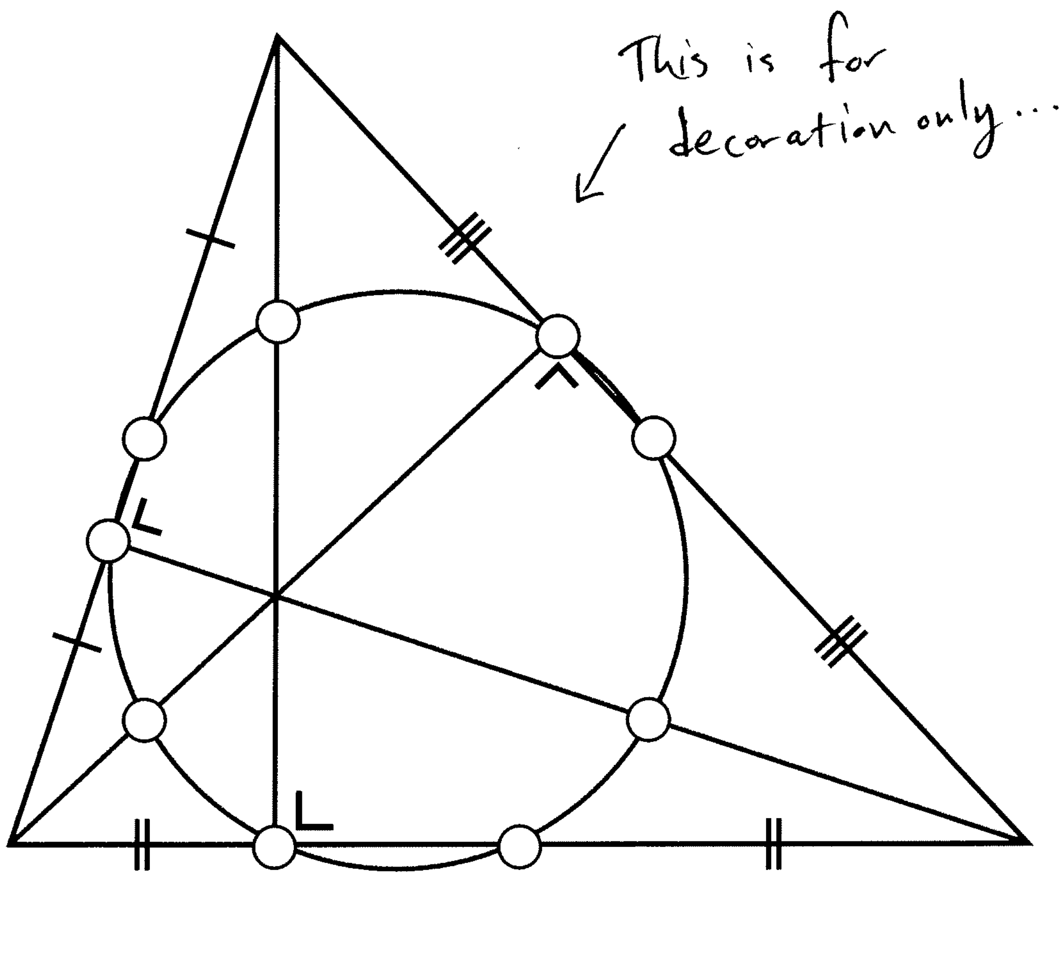
Right circle, left circle...

- Mr. Miyagi

Math 383 Modern Geometry
Spring 2010 Exam 1
Professor: Shawn Rafalski

Write your name on this exam right now. Your work on this exam is to be your work alone. You have one hour to finish. Explain your answers clearly, and *show your work*. This exam has 12 pages, and the questions are worth a total of 100 points (not including bonus points, which are clearly marked). Only work on bonus questions after you have tried all the regular questions. Breathe regularly, and good luck!!



1. (6 points each) Each statement below is either true or else it can be made true with a very minor correction. Say whether each statement is true or false, and correct each false statement to make it true (if you'd like, you can do this by crossing out parts of the statement, writing in parts, etc).

   (a) If a right triangle is inscribed in a circle, then one side of the triangle must be a diameter of the circle.

   True by Star tek.

   G centroid

   ~~I~~ O H

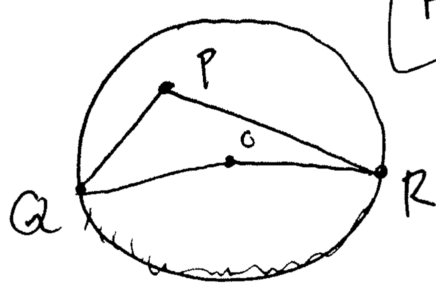
   (b) The ~~incenter~~, the circumcenter, and the orthocenter of a triangle all lie on a single line.

   False

   on

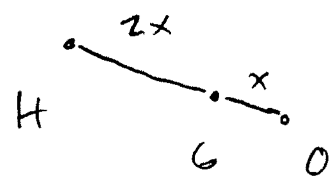
   (c) If P is a point ~~in the interior of~~ the circle Γ, and Q and R are two points on Γ, then the angle $\angle QPR$ is half the measure of the arc on Γ from Q to R.

   False.



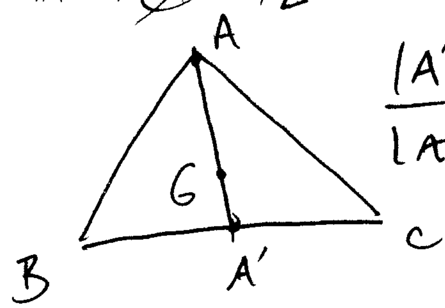
(d) In a triangle, the distance from the orthocenter to the centroid is ~~half~~ 2/3 the distance from the orthocenter to the circumcenter.

$$\frac{|HG|}{|HO|} = \frac{2}{3}$$



False

(e) If G is the centroid of the triangle ΔABC, and if A' is the midpoint of BC, then $|A'G|/|AG| = \cancel{2/3}$. 1/2



$$\frac{|A'G|}{|AG|} = \frac{1}{2}$$

False

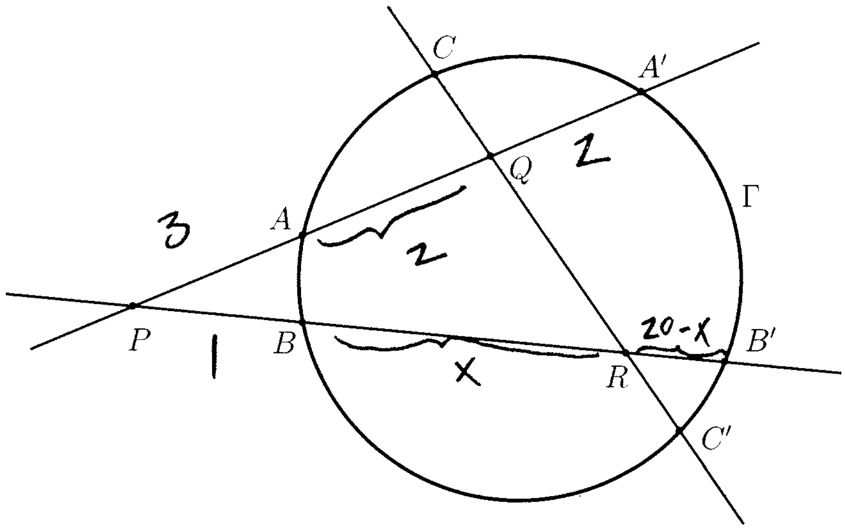
(f) The incenter of a triangle is the intersection of its angle bisectors.

True

(g) The ~~ortho~~ circum center of a triangle is the intersection of the perpendicular bisectors of the sides of the triangle.

False

(or change "perpendicular bisectors of the sides" to "the altitudes")



2. (12 points) Consider the configuration of lines and points, pictured above. Let Γ denote the circle. Suppose that $|PA| = 3$, $|PB| = 1$ and $|QA'| = 2$, and suppose that $\Pi(Q) = 4$ and $\Pi(R) = 36$ (where $\Pi(\cdot)$ denotes the power of a point with respect to Γ). If you know that $|BR'| < |BR|$, then compute $|PR|$.

$\Pi(P) = 3 \cdot 7 = 21$.

So $|PB'| = 20$.

$x \cdot (20 - x) = 36 \rightarrow 20x - x^2 = 36$

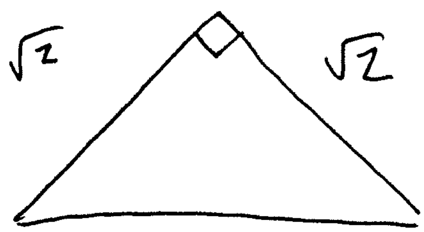
$x^2 - 20x + 36 = 0$

$(x - 18)(x - 2) = 0$

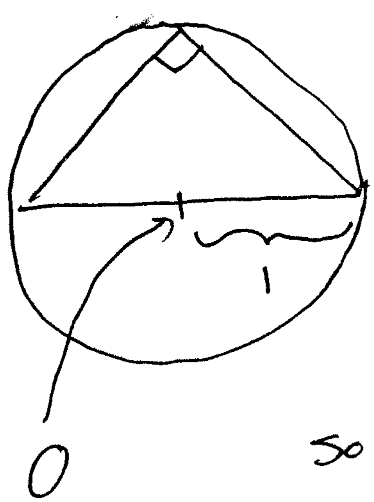
So $|BR| = 18$, so $|PR| = 1 + 18 = \boxed{19}$

3. (9 points each) Compute the area of ***two of the following three*** circles.

(a) The circumcircle of a right-angled isosceles triangle whose two equal sides each have length $\sqrt{2}$.

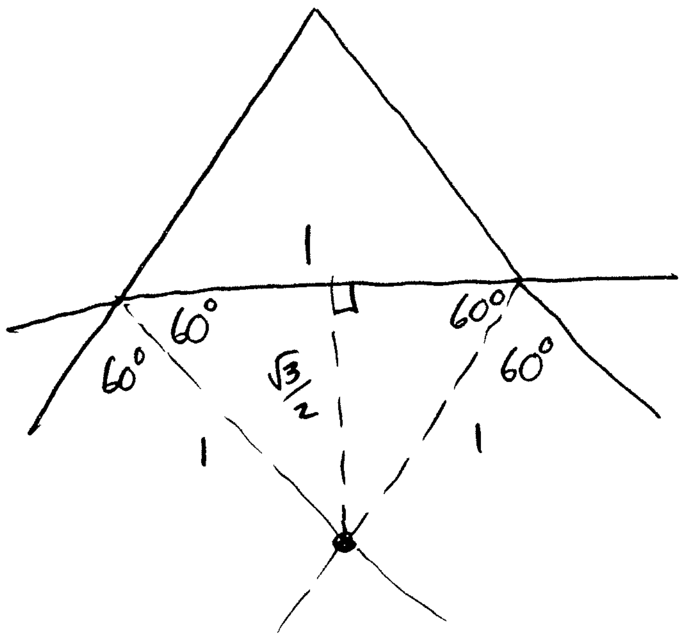


$\sqrt{2+2} = 2$



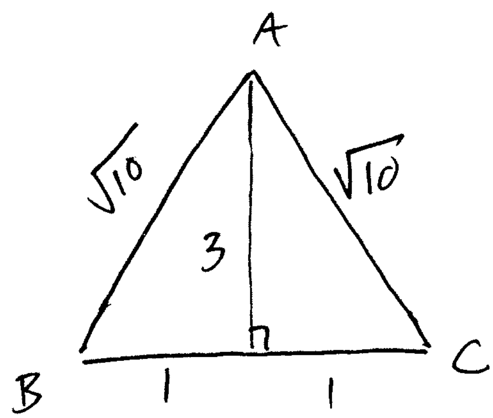
so Area (circumcircle) $= \pi \cdot 1^2 = \boxed{\pi}$

(b) The excircle of an equilateral triangle of side length 1.



$$\pi \cdot \left(\frac{\sqrt{3}}{2}\right)^2 = \boxed{\frac{3}{4}\pi}$$

(c) The incircle of an isosceles triangle ABC with $|AB| = |AC|$ and base length $|BC| = 2$ and height 3. (The height is measured with BC as the base.)



$$s = \frac{2\sqrt{10}+2}{2} = \sqrt{10}+1$$

$$\text{Area} = \frac{1}{2}\cdot 2 \cdot 3 = 3.$$

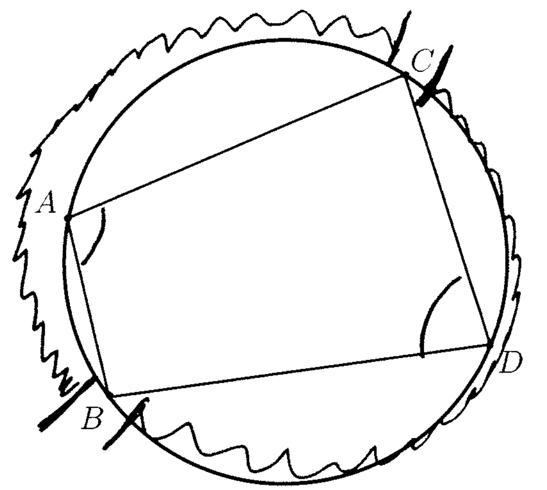
so $3 = s \cdot r = (\sqrt{10}+1)\, r$

$$\downarrow$$

$$\frac{3}{\sqrt{10}+1} = r$$

$$\downarrow$$

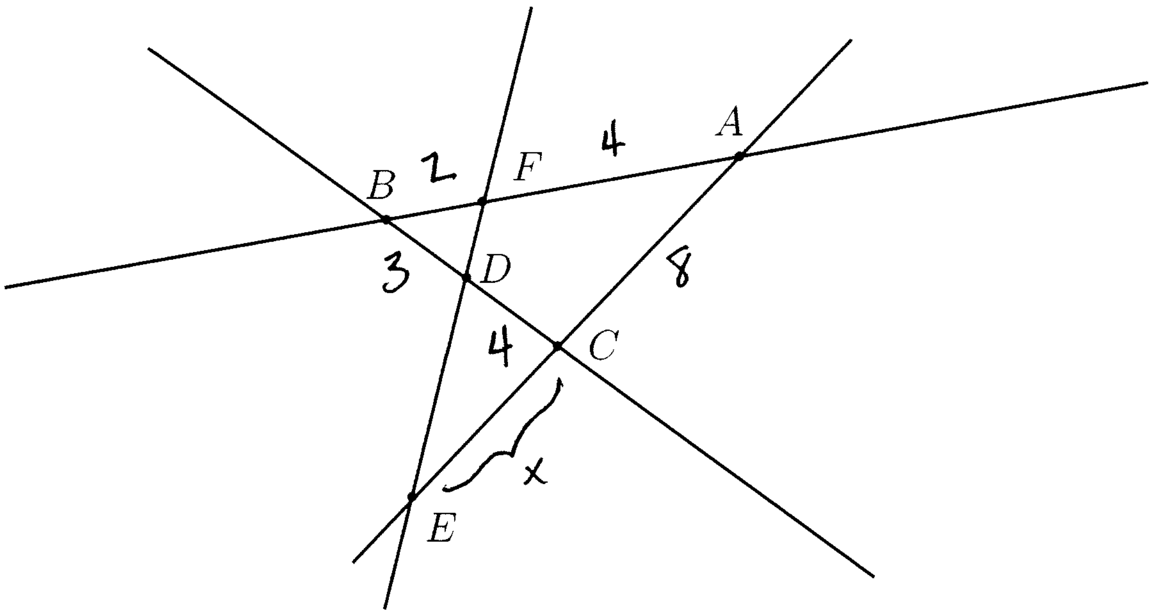
$$\text{Area (Incircle)} = \pi\left(\frac{3}{\sqrt{10}+1}\right)^2$$



4. (8 points) Let $ABCD$ be a quadrilateral inscribed in a circle, as in the above figure. Use the Star Trek lemma to find the sum of the two opposite interior angles at A and D. Explain your reasoning.

The inscribed angles at A & D subtend two arcs whose total length is 360°. So by the ST lemma, the angle sum is $\frac{360^\circ}{2} = \boxed{180^\circ}$.

5. Consider the following configuration of lines in the plane, and answer the questions below.



(a) (12 points) If $|AF| = 4$, $|FB| = 2$, $|BD| = 3$, $|DC| = 4$, and $|CA| = 8$, then calculate the length $|CE|$. (If you get stuck on this, I will give you partial credit for providing the complete and correct statement of either Menelaus' Theorem or Ceva's Theorem.)

$$\left|\frac{4}{2}\cdot\frac{3}{4}\cdot\frac{x}{x+8}\right| = |-1| = 1$$

$$\downarrow$$

$$\frac{x}{x+8} = \frac{2}{3} \longrightarrow \boxed{x = 16}$$

(b) (8 points) Calculate $\cos \angle BAC$.

$$7^2 = 6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cos(\angle BAC)$$

$$\downarrow$$

$$\frac{49 - 36 - 64}{-2 \cdot 6 \cdot 8} = \cos(\angle BAC)$$

$$\boxed{51/96} = \boxed{\frac{17}{32}}$$

(c) (**Bonus 2 points**) Compute the circumradius of triangle ABC. (You can leave your answer unsimplified.)

6. (**BONUS 3 points**) Prove any **one** of the following.

   (a) The Pythagorean Theorem

   (b) The Naerogahtyp Theorem (the converse of (a))

   (c) If H is the orthocenter of ΔABC, then A is the orthocenter of ΔHBC

   (d) If P and P' are two points in the interior of Γ that are the same distance to the center of Γ, then the power of P equals the power of P'.

7. (**Bonus 1 point each**) Answer the following:

   (a) What is the name of the Google search algorithm?

   (b) According to Dr. Demers, the experimental value that Google has determined for the relevance percentage of its search algorithm (denoted by "d") is closest to which value: 0.65, 0.75, 0.85, 0.95?

   (c) In *The Simpsons* episode $Homer^3$, what does the police chief do to try and save Homer Simpson from the 3rd dimension?

   (d) What kind of store does Homer walk into at the end of the episode from the previous question?