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Math Stats 3352

Problem Set I (Solutions)

Due Jan 29

1. This is to prove Theorem 6.10 in the book.

(a) [2pt] Show that if X and Y have a bivariate normal distribution with correlation coefficient $\rho = 0$ then X and Y are independent. This is pretty easy, just use the defining formula for the bivariate normal and the definition of independence. Since it is easy do it carefully! If $\rho = 0$ then

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]} \cdot \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_Y}{\sigma_Y}\right)^2\right]} \\ &= f_X(x) \cdot f_Y(y) \end{aligned}$$

which proves independence.

(b) [4pt] Show without using the formula for the covariance of X and Y that if X and Y have a bivariate normal distribution with correlation coefficient $\rho \neq 0$ then X and Y are not independent. To show it is not independent you will need to know the marginal distributions (you do!) and the definition of independence (you do!). After that it is algebra but make sure you really know that the two sides are not equal.

Of course the result follows from $\rho = \text{Cov}(X,Y)/\sigma_X\sigma_Y$. Since we did not prove this, my rigorous sense of honor forbids me from doing it that way, but from there it is easy to argue that independence implies covariance is 0 (Thm. 4.12) which implies $\rho = 0$. It is important to note that the argument would not work in reverse for (a) because covariance = 0 does not imply independence (Example 4.17).

For the approach that does not rely on the unproven result, we need to show that when $\rho \neq 0$

$$f(x,y)\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

is not equal to

$$f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]} \cdot \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_Y}{\sigma_Y}\right)^2\right]}.$$

they don't not look equal when $\rho \neq 0$, but how can we say for sure? Well if they were equal they would be equal when, for instance, $x = \mu_X$ and $y = \mu_Y$, which I picked because it eliminates everything in the exponent. So independence would imply

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} = \frac{1}{\sqrt{2\pi}\sigma_X}\frac{1}{\sqrt{2\pi}\sigma_Y}$$

which requires that $\sqrt{1-\rho^2} = 1$ which implies $\rho = 0$. Important point: independence says that two functions of x and y are equal. For functions to be equal they have to be equal for all vaules of x and y, so to disprove independence it suffices to show inequality for one particular value of x and y (as we did). However, since we are claiming dependence for all values of $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and all values of $\rho \neq 0$, it does not suffice to show dependence for certain values of these paramaters. Its tricky, but its just quantifiers!

2. [3pt] Do problem 6.46 in the book. If we plug $\mu_X = 0$ and $\mu_Y = -1$ into the exponent

$$-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

we get

$$-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_X^2}x^2 - \frac{2\rho}{\sigma_Y\sigma_X}x(y+1) + \frac{1}{\sigma_Y^2}(y+1)^2\right]$$

which tells us

$$\begin{aligned}\frac{1}{2(1-\rho^2)\sigma_X^2} &= \frac{1}{54}\\ \frac{1}{2(1-\rho^2)\sigma_Y^2} &= \frac{2}{27}\\ \frac{\rho}{(1-\rho^2)\sigma_X\sigma_Y} &= -\frac{1}{27}.\end{aligned}$$

The ratio of the first two tells us

$$\sigma_X = 2\sigma_Y$$

and the ratio of the last two tells us

$$\rho = -1/2$$

from which we get, plugging into any of the three equations

$$\begin{aligned}\sigma_X &= 3\\ \sigma_Y &= 6.\end{aligned}$$

3. [2pt] Suppose the random variable X takes on the values $-3, -2, -1, 0, 1, 2, 3$ with equal probabilities, and the random variable Y is given by $Y = X^2 - X$. Find the pdf of Y. We just apply this function to each of these seven values of X

x	-3	-2	-1	0	1	2	3
y	12	6	2	0	0	2	6
$P(X = x)$	1/7	1/7	1/7	1/7	1/7	1/7	1/7

which combining terms with the same value of y

y	12	6	2	0
$P(Y = y)$	1/7	2/7	2/7	2/7

4. [5pt] Do problem 7.5 in the book. The algebra is tricky, but if you make sure your answer and the steps along the way give you the results in Example 7.3 when you plug in $\theta_1 = 1/3$ and $\theta_2 = 1/2$ you should be fine.

(a)

$$
\begin{aligned}
F(y) &= \int_0^y \int_0^{y-x_2} \frac{1}{\theta_1\theta_2} e^{-x_1/\theta_1 - x_2/\theta_1} dx_1\, dx_2 \\
&= \int_0^y -\frac{1}{\theta_2} e^{-x_1/\theta_1 - x_2/\theta_2}\Big|_0^{y-x_2} dx_2 \\
&= \int_0^y \frac{1}{\theta_2} e^{-x_2/\theta_2}\left[1 - e^{-y/\theta_1 + x_2/\theta_1}\right] dx_2 \\
&= \int_0^y \left[\frac{1}{\theta_2} e^{-x_2/\theta_2} - \frac{1}{\theta_2} e^{-y/\theta_1 - (\theta_1-\theta_2)x_2/\theta_2\theta_1}\right] dx_2 \\
&= -e^{-x_2/\theta_2} + \frac{\theta_1}{\theta_1 - \theta_2} e^{-y/\theta_1 - (\theta_1-\theta_2)x_2/\theta_2\theta_1}\Big|_0^y \\
&= 1 - e^{-y/\theta_2} - \frac{\theta_1}{\theta_2 - \theta_1}\left[e^{-y/\theta_1} - e^{-y/\theta_1 - (\theta_1-\theta_2)y/\theta_2\theta_1}\right] \\
&= 1 - e^{-y/\theta_2} - \frac{\theta_1}{\theta_2 - \theta_1} e^{-y/\theta_1} + \frac{\theta_1}{\theta_1 - \theta_1} e^{-y/\theta_2} \\
&= 1 + \frac{\theta_2}{\theta_1 - \theta_2} e^{-y/\theta_2} - \frac{\theta_1}{\theta_1 - \theta_2} e^{-y/\theta_1}.
\end{aligned}
$$

You have to keep militantly simplifying everything you can as you go or it becomes a monster that will eat your home town. Taking the derivative yields

$$
f(y) = \frac{1}{\theta_1 - \theta_2}\left[e^{-y/\theta_1} - e^{-y/\theta_2}\right].
$$

(b) The above argument breaks down when $\theta_1 = \theta_2$ because we divide by $\theta_1 - \theta_2$.

So let's imitate it when they are equal and see what we get.

$$\begin{aligned}
F(y) &= \int_0^y \int_0^{y-x_2} \frac{1}{\theta^2} e^{-x_1/\theta - x_2/\theta} dx_1\, dx_2 \\
&= \int_0^y -\frac{1}{\theta} e^{-x_1/\theta - x_2/\theta} \Big|_0^{y-x_2} dx_2 \\
&= \int_0^y \left[\frac{1}{\theta} e^{-x_2/\theta} - \frac{1}{\theta} e^{-y/\theta} \right] dx_2 \\
&= -e^{-x_2/\theta} - \frac{x_2}{\theta} e^{-y/\theta} \Big|_0^y \\
&= -\frac{y}{\theta} e^{-y/\theta} - e^{-y/\theta} + 1
\end{aligned}$$

which upon taking the derivative gives

$$f(y) = \frac{y}{\theta^2} e^{-y/\theta} - \frac{1}{\theta} e^{-y/\theta} + \frac{1}{\theta} e^{-y/\theta} = \frac{y}{\theta^2} e^{-y/\theta}.$$

Golly, how do we know this is right? Well it should be the limit of the answer to (a) as θ_1 goes to θ_2. This limit requires L'Hopital!! Take the derivative of the top and bottom with respect to θ_1 and then set it equal to $\theta_2 = \theta$

$$\lim_{\theta_1 \to \theta_2} \frac{e^{-y/\theta_1} - e^{-y/\theta_2}}{\theta_1 - \theta_2} = \frac{\frac{y}{\theta_1^2} e^{-y/\theta_1}}{1} \to \frac{y}{\theta^2} e^{-y/\theta}$$

Who doesn't think that's cool!

5. [4pt] Suppose that X has a continuous uniform distribution on the interval $[0, 1]$. Construct a random variable $Y = r(X)$ whose pdf is

$$g(y) = \begin{cases} \frac{3}{8} y^2 & 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

You will need to find the function r by following the process of 7.2 backwards. It starts by finding the cdf of Y. From this you will be able to see what $r^{-1}(y)$ is, and then you just have to invert.

$$G(y) = \int_0^y \frac{3}{8} y^2 dy = \frac{1}{8} y^3 \qquad \text{for } 0 < y < 2$$

(to the left of that it is zero, to the right it is 1). $G(y) = P(Y \leq y) = P(r(X) \leq y) = P(X \leq r^{-1}(y)) = F_X\left[r^{-1}(y)\right]$. Since X is uniform $f_X(x) = 1$ so $F_X(x) = x$, and thus

$$r^{-1}(y) = G(y) = \frac{1}{8} y^3 \qquad \text{for } 0 < y < 2$$

so inverting the function in the usual fashion

$$\begin{aligned}
x &= \frac{1}{8} y^3 \\
y &= \sqrt[3]{8x} = 2\sqrt[3]{x}
\end{aligned}$$

which the usual inverse calculation tells you

$$r(x) = 2\sqrt[3]{x}.$$

Out of 20 points