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Math 3352

Problem Set IV (Solutions)

Due March 23

All write-ups should be individually done, complete, neat and precise. Please tell me anyone you worked with or got help from and AI tools, websites or box you used and exactly how.

1. [2pt] Question 8.21 in Book: Use Thm. 8.11 to show, for a sample of size n from a normal population with variance σ^2, the sampling distribution of S^2 has mean σ^2 and variance $2\sigma^4/(n-1)$. Theorem 8.11 tells us that $(n-1)S^2/\sigma^2$ has a χ^2 distribution with $n-1$ degrees of freedom. That means by Corollary 6.2 it has mean of $n-1$ and variance $2(n-1)$. So

$$\begin{aligned} E((n-1)S^2/\sigma^2) &= n-1 \\ \frac{n-1}{\sigma^2}E(S^2) &= n-1 \\ E(S^2) &= \sigma^2. \end{aligned}$$

$$\begin{aligned} \mathrm{Var}((n-1)S^2/\sigma^2) &= 2(n-1) \\ \frac{(n-1)^2}{\sigma^4}\mathrm{Var}(S^2) &= n-1 \\ \mathrm{Var}(S^2) &= \frac{2\sigma^4}{n-1}. \end{aligned}$$

2. 8.4, p. 272: If $X_1, X_2, \ldots, X_n$ are independent random variables having identical Bernoulli distributions with parameter θ, then $\overline{X}$ is the *proportion* of successes in n trials, which we denote by $\hat{\Theta}$. Show that

   (a) [1pt] $E(\hat{\Theta}) = \theta$. According to Theorem 8.1 $E(\hat{\Theta}) = \mu$, which is the mean of each of these Bernoulli variables, which we know to be θ (Thm. 5.2)

   (b) [2pt] $\mathrm{Var}(\hat{\Theta}) = \theta(1-\theta)/n$. Again Thm. 8.1 says $\mathrm{Var}(\hat{\Theta}) = \mathrm{Var}(X)/n$, and Thm. 5.2 says for X Bernoulli $\mathrm{Var}(X) = \theta(1-\theta)$ so that's a wrap.

3. [3pt] Question 8.22 in Book: Show if $X_1, X_2, \ldots, X_n$ are independent random variables each satisfying a χ^2 distribution with $\nu = 1$ degrees of freedom and $Y_n = X_1 + X_2 + \cdots + X_n$, then in the limit the distribution

$$Z = \frac{Y_n/n - 1}{\sqrt{2/n}}$$

approaches the standard normal distribution as $n \to \infty$. Y_n/n is just the average $\overline{X}$. Since they satisfy χ^2 with $\nu = 1$, they each have $\mu = 1$ and $\sigma^2 = 2$ (by Cor. 6.2 as above). So $Z = (Y_n/n - 1)/(\sqrt{2}/\sqrt{n})$ is the standardized version of $\overline{X}$, which the Central Limit Theorem proved approached the standard normal as n goes to infinity (since the variables are independent).

4. 8.73, p. 292: The actual proportion of families in a certain city who own rather than rent their home is 0.70. Suppose we planned to ask 84 families in the city selected at random whether they own or rent. If we assign each response a 1 if they own and a 0 if they rent, the results represent 84 independent random variables with Bernoulli distributions (with $\theta = .7$ of course). Use Exercise 8.4 to estimate the probability that $\hat{\Theta}$ (the proportion of people *in the sample* who own) lies between 0.64 and 0.76, using

(a) [2pt] Chebychev's Theorem The distribution of $\hat{\Theta}$ has a mean of $\theta = .7$ and a standard deviation on $\sqrt{\theta(1-\theta)/n} = \sqrt{.7 \times .3/84} = .05$. So Chebychev tells us the percentage of time we fall within 1.2 standard deviations of the mean, or between 0.64 and 0.76 is at least $1 - 1/1.2^2 = .3056$.

(b) [2pt] The central limit theorem. Since $.7 \times 84$ and $.3 \times 84$ are fairly large (certainly much larger than 5) we can assume the distribution of the sum and therefore the average is roughly normal, so we can approximate the probability by the probability of a normal variable with mean .7 and s.d. .05 falling between .64 and .76, which normdist tells me is approximately .7699.

5. [2pt] 8.81, p. 293: If S_1^2 and S_2^2 are the variances of independent random samples of size $n_1 = 10$ and $n_2 = 15$ from normal populations with equal variances, find $P(S_1^2/S_2^2 < 4.03)$. We know that (assuming the common pop. variance is σ^2)

$$F = \frac{S_1^2 \sigma^2}{S_2^2 \sigma^2} = \frac{S_1^2}{S_2^2}$$

has an F distribution with $(9, 14)$ degrees of freedom so

$$P((S_1^2/S_2^2 < 4.03) = 0.9900.$$

6. [3pt] 10.5, p. 326: Given a random sample of size n from a population that has the known mean μ and the finite variance σ^2, show that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

is an unbiased estimator of σ^2.

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2\right] = \frac{1}{n}\sum_{i=1}^{n} E\left[(X_i - \mu)^2\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \sigma^2.$$

7. [3pt] I asked a simple random sample of 45 adults who watched President Trump's recent state of the union speech how many minutes they watched it for, and found an average of 32 minutes with a sample standard deviation of 21 minutes. Give a 90% confidence interval for the population mean and report it in a complete sentence.

Out of 20 points.