

Test I (Solutions)

You may use 2 pages of notes and a calculator and/or computer. When I say describe a distribution I mean identify which of our standard distributions it is and the value of whatever parameters are needed to specify it: For example the normal distribution with $\mu = 5$ and $\sigma = 3$, or the chi-squared distribution with $\nu = 4$ degrees of freedom.

1. [15pt] Suppose X and Y are independent normal variables with $\mu_X = 0$, $\mu_Y = 4$, $\sigma_X = 4$, $\sigma_Y = 3$. Recall that $W = X + Y$ and $Z = 4Y - 3X$ are bivariate normal. Describe the joint distribution of W and Z (Remember $\rho = \text{Cov}(W, Z)$).

$$\begin{aligned}\mu_W &= \mu_X + \mu_Y = 0 + 4 = 4 \\ \mu_Z &= 4\mu_Y - 3\mu_X = 12 - 0 = 12 \\ \sigma_W^2 &= \sigma_X^2 + \sigma_Y^2 = 3^2 + 4^2 = 5^2 \\ \sigma_Z^2 &= 16\sigma_Y^2 + 9\sigma_X^2 = 2 \cdot 12^2\end{aligned}$$

So $\sigma_W = 5$ and $\sigma_Z = 12\sqrt{2}$. Finally

$$\text{Cov}(W, Z) = \text{Cov}(X + Y, 4Y - 3X) = 4\text{Var}(Y) - 3\text{Var}(X) = 4 \cdot 9 - 3 \cdot 16 = -12$$

so

$$\rho = -12 / (5 \times 12\sqrt{2}) = -\frac{1}{5\sqrt{2}}$$

which is enough to specify a bivariate normal distribution.

2. [15pt] If X has a binomial distribution with $n = 3$ and $\theta = 1/3$, Find the distribution of $Y = X/(X + 1)$.

$$f(x) = \binom{3}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x}.$$

P	X	Y
$\frac{8}{27}$	0	0
$\frac{12}{27}$	1	$1/2$
$\frac{6}{27}$	2	$2/3$
$\frac{1}{27}$	3	$3/4$

3. [15pt] If X has an exponential distribution with parameter θ , Find the pdf of $Y = e^{-X}$, and then check that it meets the conditions of a pdf.

$$f_X(x) = \frac{1}{\theta} e^{-x/\theta} \quad x > 0$$

so

$$F_X(x) = \int_0^x \frac{1}{\theta} e^{-t/\theta} dt = 1 - e^{-x/\theta}$$

$$F_Y(y) = P(e^X < y) = P(X < \ln(y)) = F(\ln(y)) = 1 - e^{-\ln(y)} = 1 - 1/y$$

so

$$f_Y(y) = F'(y) = \frac{1}{y^2} \quad 1 \leq y < \infty.$$

No doubt that is positive, so check the integral is one

$$\int_1^{\infty} \frac{dy}{y^2} = -\frac{1}{y} \Big|_1^{\infty} = 0 - -1 = 1.$$

4. [15pt] The 200 prairie dogs in the prairie dog village you are studying have an average weight of 1500 grams with a standard deviation of 500 grams. If you capture 100 of them and weigh them, what are the mean and standard deviation of your \bar{X} ? For a finite population

$$\begin{aligned} \mu_{\bar{X}} &= \mu_X = 1500 \\ \sigma_{\bar{X}} &= \frac{\sigma_X}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} = \frac{500}{10} \sqrt{\frac{100}{199}} = 35.4 \end{aligned}$$

5. [10pt] Adult men's heights in America have a mean of 185 pounds and a standard deviation of 33 pounds. If 15 adult men get on an elevator rated to hold 3300 pounds, what is the probability they will exceed the capacity of the elevator? What did you have to assume about the distribution of mens weights and the way the men were selected to conclude this? If the total exceeded 3300 pounds, the average would exceed 220. The average has a mean of 185 and a standard deviation of $33/\sqrt{15} = 8.52$.so

$$p = 1 - \text{normdist}(220, 185, 8.52, 1) = .00002.$$

Assumed it was a simple random sample of all men in America, that the population was infinite (reasonable approximation because the pop is more than 20 times the sample size) and that men's heights are normally distributed (with $n = 15$ we just need it to be not very skewed).

6. [15pt] A sample of size $n = 5$ is taken from a normal population with a standard deviation of $\sigma = 5$. What is the probability the sample variance will be more than 20 but less than 35. Recall that the chi-squared distribution is helpful for describing the distribution of S^2 . We know

$$\frac{(n-1)S^2}{\sigma^2} = \frac{4S^2}{25}$$

has a *chi*² distribution with 4 degrees of freedom, so

$$20 < S^2 < 35$$

means

$$\frac{16}{5} < \frac{4S^2}{25} < \frac{28}{5}$$

or

$$P\left(\frac{16}{5} < \chi_4^2 < \frac{28}{5}\right) = \text{CHIDIST}(16/5, 4) - \text{CHIDIST}(28/5, 4) = .294.$$

7. [15pt] The number of piercings of college students is known to be normally distributed. You take a simple random sample of 16 college students and ask them how many piercings they have and get the following answers

0, 2, 3, 0, 0, 1, 2, 2, 2, 5, 4, 2, 0, 0, 1, 3.

Find \bar{X} and S^2 for this sample. If the average number μ_X of piercings among all college students is 1, what is the probability a random sample of 16 students would have an \bar{X} greater or equal to your value? If instead $\mu_X = 1.4$, what is that probability? which of those means seems like a more likely valuer given your sample. The mean of your sample is $\bar{X} = 1.69$. the sample standard deviation is 1.537.

If $\mu_X = 1$, the t -score for your result is

$$t = \frac{\bar{X} - \mu_X}{S/\sqrt{n}} = \frac{1.69 - 1}{1.537/4} = 2.75.$$

The probability of getting this t -score or higher is (15 degrees of freedom, one tailed)

$$P(t > 2.75) = \text{TDIST}(2.75, 15, 1) = .00744$$

or less than 1%.

If $\mu_X = 1.4$, the t -score for your result is

$$t = \frac{\bar{X} - \mu_X}{S/\sqrt{n}} = \frac{1.69 - 1.4}{1.537/4} = .8214.$$

The probability of getting this t -score or higher is (15 degrees of freedom, one tailed)

$$P(t > .8214) = \text{TDIST}(.8214, 15, 1) = .212$$

so it would happen 21% of the time. Our result would be very surprising in the first case and not at all unusual, so the second option is more likely.

Out of 100 points