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MATH 3352

Test I (Solutions)

You may use two pages of notes, calculator and computer. When I say identify a distribution I mean identify which of our standard distributions it is and the value of whatever parameters are needed to specify it: For example the normal distribution with $\mu = 5$ and $\sigma = 3$, or the chi-squared distribution with $\nu = 4$ degrees of freedom.

1. X has a normal distribution with mean $\mu_X = 6$ and s.d. $\sigma_X = 4$, Y has a normal distribution with mean $\mu_Y = 4$ and s.d. $\sigma_Y = 1$, and X nd Y are independent.

   (a) [4pt] Identify the distribution $W = (X - 3)/2$. It is a normal distribution with

$$\mu_W = (\mu_X - 3)/2 = (6 - 3)/2 = 1.5$$
$$\sigma_W = \sigma_X/2 = 2.$$

   (b) [6pt] Identify the distribution $Z = X - Y$. It is a normal distribution with

$$\mu_Z = \mu_X - \mu_y = 6 - 4 = 2$$
$$\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{4^2 + 1^2} = \sqrt{17}.$$

   (c) [6pt] Identify the distribution of

$$(x - 6)^2/16 + (Y - 4)^2.$$

   It is the sum of the squares of two independent standard normals, so it has a Chi-Squared distribution with 2 degrees of freedom.

2. Recall a continuous uniform distribution on all real numbers from α to β has pdf $f(x) = 1/(\beta - \alpha)$ for $\alpha < x < \beta$, moment generating function

$$M_X(t) = \frac{e^{\beta t} - e^{\alpha t}}{\beta - \alpha},$$

   a mean $(\alpha + \beta)/2$ and a variance $(\beta - \alpha)^2/12$. You have your calculator randomly generate two random numbers X and Y, each between 0 and 1 uniformly, and declare the variable $Z = X + Y$ to be the sum of these.

   (a) [5pt] What are the mean and standard deviation of X?

$$\mu = (0 + 1)/2 = 1/2 \qquad \sigma = \sqrt{(1 - 0)^2/12} = 1/2\sqrt{3} = 0.289.$$

   (b) [9pt] What are the mean and standard deviation of Z?

$$\mu_Z = \mu_X + \mu_Y = 1/2 + 1/2 = 1. \qquad \sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{1/12 + 1/12} = \sqrt{1/6} = .408.$$

(c) [5pt] What were you implicitly assuming about X and Y in your answer to the last question? Independence!

(d) [10pt] What is the moment generating function of Z?

$$M_Z(t) = M_X(t)M_Y(t) = \frac{e^t - 1}{1}\frac{e^t - 1}{1} = \left(e^t - 1\right)^2.$$

(e) [10pt] Find the probability that $Z < 1/2$. Draw the picture of the region before you set up the integral. It is a triangle, so of course we can find the area without doing an integral but we love to integrate, so

$$\begin{aligned} P(Z < 1/2) &= \int_0^{1/2}\int_0^{1/2-y} 1dx\,dy \\ &= \int_0^{1/2} \frac{1}{2} - ydy \\ &= \frac{1}{2}y - \frac{y^2}{2}\bigg|_0^{1/2} \\ &= \frac{1}{8}. \end{aligned}$$

3. [15pt] Suppose X has an exponential distribution with $\theta = 4$. What is the pdf for $Y = X^2$? (notice the function $y = x^2$ is invertible on $x \geq 0$) Using the transformation technique, with $X = \sqrt{Y}$

$$\begin{aligned} f_Y(y) = f_X(\sqrt{y})\left(\sqrt{y}\right)' &= \frac{1}{4}e^{-\frac{1}{4}\sqrt{y}}\left(y^{1/2}\right)' \\ &= \frac{1}{4}e^{-\frac{1}{4}\sqrt{y}}\frac{1}{2}y^{-1/2} = \frac{1}{8\sqrt{y}}e^{-\frac{1}{4}\sqrt{y}}. \end{aligned}$$

4. [15pt] The 100 stars who attended the Grammies wore outfits that cost an average of \$47,000 with a standard deviation of \$12,000. If you randomly selected 40 stars, what is the probability that the average value of all their outifts was between \$40,000 and \$50,000? Notice the size of the population! We use the finite population correction

$$\begin{aligned} \mu_{\overline{X}} &= \mu_X = 47 \\ \sigma_{\overline{X}} &= \frac{\sigma_X}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \frac{12}{\sqrt{40}}\sqrt{\frac{60}{99}} = 1.48. \end{aligned}$$

Of course the central limit theorem still applies, so with $n \geq 40$ we can assume $\overline{X}$ is normally distributed so the probability is

$$\text{NORMDIST}(50, 47, 1.48, 1) - \text{NORMDIST}(40, 47, 1.48, 1) = 97.9\%.$$

Wait, why was I allowed to divide everything by 1000? Think about it!

5. [15pt] Adult American women's heights are approximately normal with a mean of 64.5 inches and a standard deviation of 3 inches. If you take a sample of 10 women and compute the sample standard deviation, what is the probability it will be between 2.5 and 3.5? We know

$$\frac{(n-1)S^2}{\sigma^2} = \frac{9S^2}{9} = S^2$$

has a χ^2 distribution with 9 degrees of freedom, so converting the S values above to S^2

$$P(6.25 < \chi^2_9 < 12.25) = \text{CHIDIST}(2.25, 9) - \text{CHIDIST}(12.25, 9) = 51.5\%.$$

Out of 100 points