This paper deals with the nature of the sampling errors which may occur in measurement of area by a square pattern 'dot planimeter'. The important conclusion is that the relative error of measurement is small if the count of points is of the order of 100 *per parcel. The authors recommend the use of different sizes of point pattern for the measurement of different sizes of parcel, rather than the use of a single overlay for the measurement of every parcel on a given map. The theoretical work is supported by some results of measurements made on the McKay Pattern Map using the MK Area Calculator.*

The Accuracy of Area Measurement by Point Counting Techniques

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Introduction

The technique of measuring areas on maps and plans by counting the number of points on a transparent overlay which occur within the boundary of an enclosed parcel has now been used for about thirty years. We believe that this sampling technique was first used by Abell[1] for area measurement in forestry work. Although it can be used in other branches of the natural sciences it appears to have been used less often than some of the other methods which are available to the map user. There are possibly three reasons for the relatively small use of the method.

In the first place it is still frequently confused with the method of area measurement by counting the number of squares and fractions of square which coincide with a parcel when a transparent square grid is placed over it. This 'Method of Squares' is what most students are taught and it appears superficially to be identical with counting points. However point counting techniques differ from counting squares because no attempt is made to estimate fractional parts round the perimeter of the parcel. Hence the total count of points will always be an integer value. The apparent loss of accuracy resulting from this approximation is compensated for by treating it as a form of statistical sampling and therefore in point counting techniques it is necessary to repeat the measuring process several times using the overlay in different random positions on the map.

Secondly, counting is a repetitive and tedious kind of work. It is therefore prone to the generation of gross errors unless very careful controls are included to ensure that the operator counts each point once and does not forget which points on the overlay have been counted or forget the total reached. Manual methods of control, such as making intermediate booking entries for every 10, 50 or 100 points counted inevitably increase the time which is needed to accomplish the measurement. There are now several different kinds of instrument which can do the counting automatically and which can be used to mark the overlay so that the risk of getting lost is more or less eliminated. These instruments have been described elsewhere.[2,3]

The third reason why point counting techniques appear to have been little used is that there has never been a proper attempt to investigate the theoretical accuracy of the methods. A few authors[4,5,6] have attempted partial explanation of error propagation, but none of these can be regarded as being adequate. In particular we ought to know how many points should be counted in order to attain some predetermined order of accuracy.

We know intuitively that if the area of the parcel is to be determined from counting a few widely-spaced points then the accuracy of the measurement may be lower than if it is based upon the count of many points. If however, the density of points is increased, then the time required to execute the count will also increase.

The increase in point density follows a geometrical progression so that the situation is soon reached when area measurement by counting points becomes slower than making the measurements by planimeter[2]. Existing knowledge about the optimum separation of points needed to make reliable measurements economically is almost wholly empirical. Most people have been content to construct an overlay by tracing points of intersection from graph paper and have therefore used a square pattern of points which are separated by simple multiples or fractions of the centimetre or inch. They have, by trial and error, found a separation of the points which appears to provide sufficient accuracy without making the job of counting too slow and laborious.

The purpose of the present paper is to attempt an evaluation of the theoretical accuracy of area measurement by counting.

The Nature of the Grid Overlay and the Techniques of Measurement

There are two different techniques of measuring area by counting points. The method which has been briefly outlined in the Introduction makes use of a regular pattern of points on the overlay. This is the method which is most commonly used and is the method which we will investigate. The second method involves the use of a random pattern of points on the overlay. It has been used less frequently but appears to be a useful technique for sampling the areas occupied by several different thematic qualities or attributes on maps or aerial photographs. For example, Stobbs[7] has described the application of this technique in evaluating the areas of different classes of land use in his investigation of Land Classification in Malawi.

If we place an overlay which shows a regular pattern of points over the parcel to be measured in a series of random positions and for every application of the overlay we make a count of $n_1, n_2, n_3 \ldots n_k$ points then the area of the parcel, S, will be

$$S = \bar{n}.s \qquad \ldots\ldots\ldots\ldots(1)$$

where $\bar{n}$ is the arithmetic mean of the points counted and s is the area of the *unit cell* formed by the pattern of points.

The pattern may usually be classed as being one of the three basic geometrical patterns which are labelled I, II and III in *Figure* 1. Pattern III is less commonly used than the other two but it has been described.[4]

In pattern I the points lie at the intersections of two families of equidistant parallel straight lines which make an angle of 60° with one another. This pattern may be interpreted equally well as a series of points situated at the vertices of equilateral triangles having side length d_1, at the centres of regular hexagons with apothem of length $\frac{1}{2}d_1$, at the centres of tangential circles with diameter d_1, or at the centres of rhombi whose smaller diagonals are equal to the side of the rhombus, d_1. *Figure* 1, I also demonstrates that every point in the pattern will be equidistant from six neighbouring points. If we take as our unit cell the hexagon which has each point on the overlay as its centre then the area of the unit cell may be expressed as:

$$s_1 = \frac{\sqrt{3}}{2}\, d_1^2 \approx 0.8660\ d_1^2 \qquad \ldots\ldots\ldots\ldots(2)$$

For some reason which is not explained, Bocharov[4] gives different formulae for each of the identical cases comprising the triangular, hexagonal, circular and rhomboidal patterns.

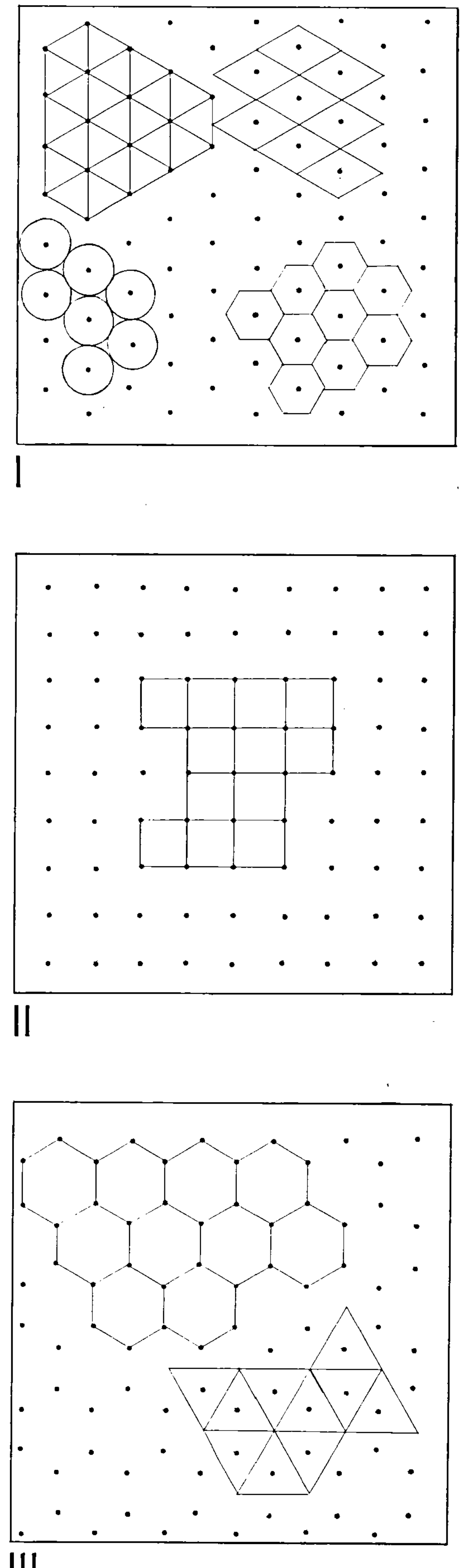

Figure 1.

Pattern II is formed by points which lie at the corners of adjacent squares. If we denote the distance between the points, measured along the rows or columns of them as d_2, then the corresponding expression for the area of the unit cell will be:

$$s_2 = d_2^2 \qquad \text{...........(3)}$$

We may note that only four points lie at the shortest distance d_2 from any given point.

The third pattern (III) is composed of points which are situated at the centres of contiguous equilateral triangles with side length $\sqrt{3}.d_3$, or at the vertices of hexagons which have side length d_3. This pattern is derived from I if every alternate horizontal row of points is excluded from it. The area of the unit cell is now

$$s_3 = \frac{3\sqrt{3}}{4}\, d_3^2 \approx 1.2990\, d_3^2 \qquad \text{...........(4)}$$

and the number of points which are of equal shortest distance, d_3, from any given point is now only three.

We may compare equations (2), (3) and (4) to find the relationship which exists between the points in each system if each cell has equal area.

Thus

$$0.9306\, d_1 \approx \frac{\sqrt[4]{12}}{2}\, d_1 = d_2 = \frac{\sqrt[4]{27}}{2}\, d_3 \approx 1.1398\, d_3 \qquad \text{...........(5)}$$

A variety of different criteria may be used to assess the uniformity of distribution of points in each pattern. One of these has already been mentioned, namely the number of points which are of minimum distance, d, from any given point. For the three patterns illustrated these represent the sequence 6, 4, 3. A second criterion is the ratio between the area occupied by the unit cell of each system. In the three patterns described these represent the ratio 0.5: 1: 2.

Both of these criteria suggest that the first pattern is preferable to the second and that both of these are better than the third. This conclusion is in agreement with most other writers on the subject.

Possible Influence of Systematic Error

We must also consider the possibility that the pattern of points on the overlay will influence the count obtained in some systematic fashion. This has already been argued by Köppke[5] in his illustration of the relationship between a long, narrow rectangular parcel and a square pattern of points. It cannot be denied that errors of measurement may arise in the single application of the overlay over a small parcel. If the parcel lies entirely between the rows and columns of points then the count $n=0$. On the other hand if the parcel coincides with one of the rows or columns of points then a very large count may be obtained. This is precisely the reason why we have to regard area measurement by this means as a technique of statistical sampling. One of us[2] has already demonstrated that Köppke's arguments become invalid if we make repeated counts with the overlay set in a succession of different random positions.

In the general case of measuring many parcels having varying sizes and different shapes it would be unreasonable to expect that significant systematic errors would arise if the overlay was applied in a truly random fashion. However it is possible that the shape of the overlay itself may inhibit random positioning.

The different positions taken by the overlay will differ from one another by a linear displacement of the point pattern and by a rotation of the pattern. Since the overlay is usually constructed on a rectangular piece of plastic it is conceivable that the operator will tend to place this with the sides more or less parallel to the map edge or the edge of the drawing board. This is especially true if the overlay is large. Then the overlay will not be rotated through any random angle, but only a small angle, θ, or through multiples of $(\pi/2 \pm \theta)$. We consider that an overlay constructed upon a circular disc of plastic facilitates the random angular setting and that the result of one measurement would not influence another measurement.

A rather different kind of systematic error occurs in practical measurement if the separation of the points differs from the nominal value of d accorded to a particular pattern and therefore the area of each unit cell is greater or less than the value of s employed to convert from number to area measure. This may arise from errors of construction, especially if a piece of graph paper is used as the basis for a square pattern without ascertaining whether it has the correct dimensions, or it may arise from deformation of the overlay in use and storage. Usually the overlay is reproduced on plastic which is subject to dimensional changes with variations in temperature, humidity, age and use. An overlay which is used for area measurement will receive much handling and may also require periodic cleaning. It is unreasonable to expect that any kind of plastic could withstand such treatment without undergoing some dimensional changes. Hence we would expect to encounter systematic errors if we always used the original value of d in the conversion factor. The only adequate method of eliminating this kind of systematic error is to make periodic tests of the dimensions of the overlay, either by checking d by linear measurement or by measuring a series of test parcels of known areas and calculating the conversion factor which best removes any bias from the errors of these test measurements.

The Elimination of Gross Errors and the Speed of Counting

We have already noted in the Introduction that an important disadvantage of point counting techniques is the ease with which gross errors may be introduced. One of us has already described some of the modern mechanical aids to counting[2] so that it is not necessary to repeat here in detail the relative merits of each instrument. However we cannot ignore their description entirely and since the speed of execution of each count depends upon the kind of equipment used, we may mention the two most important instruments with reference to their operating speeds.

The more successful of the two instruments tried by us is the British-made equipment called the Markounter. This was originally intended for counting bacterial colonies and for similar biological purposes but it can be used with any kind of point overlay without modification. The instrument comprises a pen holder containing an electrical switch which is connected to a suitable counting mechanism. The holder is fitted with a pen or pencil to mark the overlay and the switch is operated by the pressure of the pen tip as it makes the mark. The pen must be raised and lowered every time one point is counted and because the existing pen holder is rather heavy it is tiring to use for large counts. Because the movements of the hand are discrete and repetitive movements it is difficult to count at a speed greater than 3 points per second. It is unlikely if many operators could maintain a counting speed faster than about $1\frac{1}{2}$ points per second for prolonged measurements.

The more interesting, but less satisfactory, instrument is the American *MK Area Calculator*. This also comprises an electrical counting device, but the electrical impulse which operates it is now made by the contact of a pencil with a conductor on the overlay. The overlay comprises a family of parallel equidistant copper strips in one direction and a family of printed guide lines perpendicular to them. The pencil is drawn along the guide lines and each time the graphite point makes contact with a copper strip the count is increased by one unit. Hence the procedure is exactly equivalent to counting a square pattern of points. The method possesses the operating advantage that the movements of the pencil are continuous drawing motions rather than discrete pointings. Therefore it is less tiring to use for long periods. However it is no faster in use than the Markounter and far less reliable. The major disadvantage of the equipment results from an inability of the counting mechanism to accept signals in quick succession. If the pencil is moved too quickly not every contact made by the pencil will be recorded on the counter and the total recorded for the parcel will be lower than it ought to be. Practical experience with the instrument suggests that it cannot be used at speeds much faster than 2 contacts per second if the risk of negative gross errors is to be avoided. The time which is required to erase the pencil lines from the overlay is an additional factor which reduces the effective operating speed of the MK Area Calculator. This does not arise in the use of the Markounter because it is possible to place a disposable piece of transparent material over the overlay which will be marked by the pen

The Analysis of Theoretical Accuracy

We propose to investigate the accuracy of area measurement by counting in terms of the square pattern of points because this is geometrically simpler to analyse than are the others. However we believe that our conclusions are substantially valid for any regular geometrical pattern and may be applied to the other arrangements of points without introducing any additional assumptions which might significantly influence them.

We assume a square pattern of points which are separated by a unit distance $d_2=1$. We will express all linear dimensions in terms of d_2 and all measurements of area in terms of the area of the square cell with sides of equal unity which is formed by four neighbouring points on the overlay.

The fundamental principle of the technique demands that we place the overlay in a random position, that we count those points which lie within the perimeter of the parcel and exclude those points lying beyond it. The count will therefore comprise discrete integer values because the technique does not allow for fractional allocation of points within the total. This implies that we must assume that each point on the overlay satisfies the Euclidean definition of possessing position but no magnitude. However we must be able to see the points in order to count them so that they must be drawn as dots or as the intersections of the families of lines which generate the desired pattern. Similarly the parcel boundary must have finite width for us to be able to see it. In practice, therefore, we may expect that certain points will overlap the perimeter of the parcel which is being measured. Then the operator must make a conscious and subjective decision whether he should count a point or not. In the theoretical analysis we assume that a point will always lie either inside or outside the perimeter of the parcel and that no overlap will occur.

If we move the overlay with respect to the parcel to be measured, this movement may comprise a linear displacement of the overlay, a rotation of it, or, more generally, some combination of both movements. Each of these movements will cause a different combination of points to coincide with the parcel and our investigation is to endeavour to discover how this happens. To do this it will be useful to isolate the displacements from the rotations.

In order to study displacement without rotation we may study the behaviour of a circle, of radius R, to be measured by means of a square pattern of points. Because of the symmetry of the circle about its centre it may be rotated through any angle without influencing the number of points counted.

Because of the symmetry of the overlay large displacements will have no significance. Consequently we may restrict our analysis to those displacements of the centre of the circle as it is moved about one-quarter of the unit cell, such as the square A of *Figure* 2. Any larger displacement of the circle with respect to the overlay will merely repeat one of the conditions which will arise as the centre is moved within this small area, which we will term the *central square.* We will show later that, in fact, we need only consider the displacements within one-eighth part of the unit cell, that is to say, within one-half of the central square divided by one of its diagonals. However the symmetry upon which this conclusion is based is not immediately obvious so we will develop the arguments initially with reference to the whole of the central square.

If the area of the circle is appreciably larger than the area of the unit cell and if the centre of the circle is moved about the central square, then some points on the overlay will always coincide with the circle. Other points will lie within the circle when the centre is in one position but will lie outside it when the centre is elsewhere. We will distinguish these sets of points as either *permanently covering* or *temporarily covering* the circle respectively.

In order to analyse the relationship between these two sets it is convenient to introduce a system of coordinate reference for the rows and columns of points on the overlay.

We introduce the (ν, η) system which is illustrated in *Figure* 2 and note that the point (ν_0, η_0) refers to the bottom left-hand corner of the central square.

It is not difficult to determine the limits within which the value of ν will lie for a circle of radius R.

Clearly

$$-R<\nu<R+0.5 \qquad \text{...........(6)}$$

If we consider the vertical columns of points for which $\nu>0$, the number of points, η_i, which permanently cover the circle will lie within the limits

$$-\sqrt{R^2-\nu^2}+0.5\leqslant\eta_i\leqslant\sqrt{R^2-\nu^2} \qquad \text{...........(7)}$$

and the number of points which will temporarily cover the circle will lie within the limits

$$\left.\begin{array}{l}\sqrt{R^2-\nu^2}<\eta_i'<\sqrt{R^2-(\nu-0.5)^2}+0.5\\ \text{and}\\ -\sqrt{R^2-(\nu-0.5^2)}<\eta_i''<-\sqrt{R^2-\nu^2}+0.5\end{array}\right\} \qquad \text{...........(8)}$$

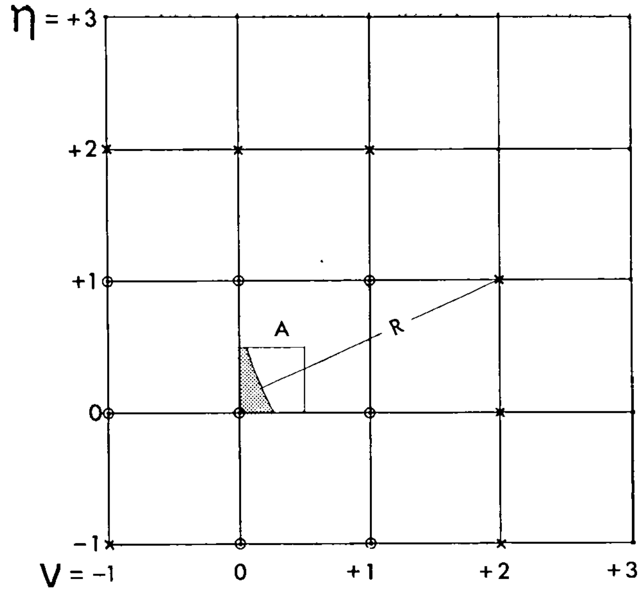

Figure 2.

Where $\nu=0$, the expression for η_i will have the form

$$-\sqrt{R^2-0.5^2}+0.5\leqslant\eta_i\leqslant\sqrt{R^2-0.5^2} \qquad \text{...........(9)}$$

and the corresponding expressions for η_i' and η_i'' will be

$$\left.\begin{aligned}&\sqrt{R^2-0.5^2}<\eta_i'<R+0.5\\ \text{and}\quad &-R<\eta_i''<-\sqrt{R^2-0.5^2}+0.5\end{aligned}\right\} \qquad \text{...........(10)}$$

Where $\nu<0$ we obtain for η_i

$$-\sqrt{R^2-(\nu-0.5)^2}+0.5\leqslant\eta_i\leqslant\sqrt{R^2-(\nu+0.5)^2} \qquad \text{............(11)}$$

with the expressions for η_i' and η_i

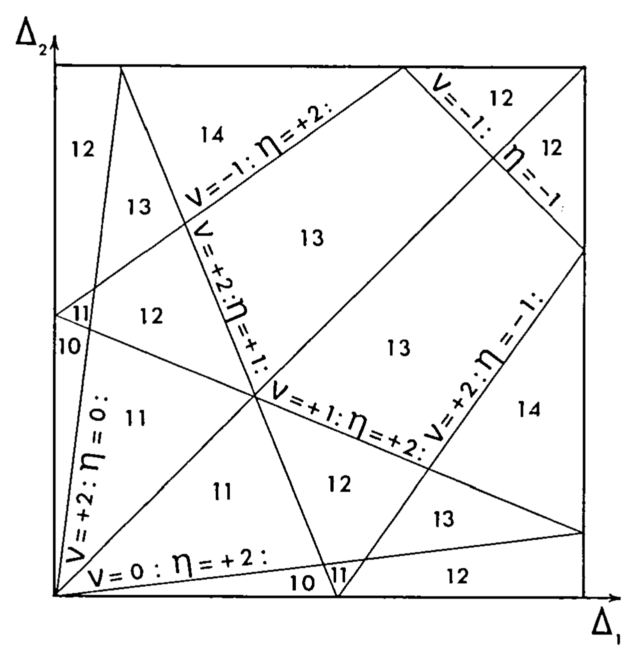

Figure 3.

$$\left.\begin{aligned}&\sqrt{R^2-(\nu+0.5)^2}<\eta_i'<\sqrt{R^2-\nu^2}+0.5\\ \text{and}\quad &-\sqrt{R^2-\nu^2}<\eta_i''<-\sqrt{R^2-(\nu-0.5)^2}+0.5\end{aligned}\right\} \qquad \text{......(12)}$$

In order to illustrate this numerically we will take the case where $R=2$. In this instance the value of ν will lie within the range -2 to $+2.5$ (expression 6). In other words it will take the integer values -1, $0+1$ and $+2$. Substituting in expressions (7) and (8) for the positive values of ν; in expressions (9) and (10) for $\nu=0$ and in (11) and (12) for the negative values of ν we find a series of values for the number of points which permanently cover the circle, η_i, and for the number of points which temporarily cover the circle, η_i', η_i''.

There will be eight points which permanently cover the circle, namely:

$\nu=+1$	$\eta_i=+1.\ 0,\ -1$
$\nu=0$	$\eta_i=+1,\ 0,\ -1$
$\nu=-1$	$\eta_i=+1,\ 0$

There will be seven points which temporarily cover the circle, namely:

$\nu=+2$	$\eta_i',\ \eta_i''=+1,\ 0,\ -1$
$\nu=+1$	$\eta_i',\ \eta_i''=+2$
$\nu=0$	$\eta_i',\ \eta_i''=+2$
$\nu=-1$	$\eta_i',\ \eta_i''=+2,\ -1$

If the total number of points is small, as in this example, the analysis may be done graphically as follows. From each of the points of the overlay which might be included in the count we draw a short arc of radius R in the vicinity of the central square. If this arc lies between the point on the overlay and the central square then that point will never coincide with the circle wherever its centre is situated in the central square. If the arc intersects the central square, then that point will temporarily cover the circle as its centre occupies different positions. If the arc lies beyond the central square then that point will always coincide with the circle, or permanently cover it, irrespective of the position of the centre within the central square. In *Figure* 2 we have indicated the points which permanently cover the circle by dots and those points which temporarily cover the circle by crosses. Consider, now, the point with coordinates $\nu=+2$, $\eta=+1$. Clearly this point will lie within the circle whenever the centre of the circle lies within the unshaded part of the central square but it will not be counted if the centre of the circle occupies a position within the shaded area. To demonstrate the probability with which a particular count of points may occur we ought to know the relative areas of each of these portions of the central square. This can be done quite easily graphically for a simple example such as our case for $R=2$.

If we redraw the central square at a suitably enlarged scale then we may calculate the points at which the various arcs intersect the sides of the square and plot them in the diagram illustrated by *Figure* 3. We regard the points of intersection with the sides of the central square as the cartesian coordinates $\triangle_1$ and $\triangle_2$ and we may find the points of intersection of any arc with these axes from the two expressions

$$\left.\begin{aligned}\triangle_1 &= \nu \pm \sqrt{-\triangle_2^2 + 2\eta\triangle_2 + R^2 - \eta^2} \\ \triangle_2 &= \eta \pm \sqrt{-\triangle_1^2 + 2\nu\triangle_1 + R^2 - \nu^2}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(13)$$

The positive square root will be used for negative values of ν and η and conversely. Having obtained the points of intersection of each arc with the sides of the central square and knowing the radius of the arc, we may now construct it with the aid of railway curves. Then we may measure the area of each figure which is composed of these arcs and the sides of the central square by planimeter. In *Figure* 3 we have indicated the number of points which would be counted for any application of a square overlay with $d_2=1$ over a circle with radius $R=2$ if the centre of the circle occupied that particular part of the central square. For example, the portion situated in the upper right-hand corner of the square is situated at a distance less than R from four points which temporarily cover the circle. In addition there are eight points which permanently cover the circle so that when the centre of the circle lies in that particular part of the central square the total count will be 12 points. For convenience of visual control, especially in the analytical method to be described, it is useful to indicate on each arc the coordinates of the point on the overlay to which it refers. This has been done in *Figure* 3. We may see in *Figure* 3 that the subdivision of the central square by the arcs is symmetrical about an axis formed by the diagonal of the square. This means that the entire analysis of displacement may be carried out with reference to the random location of the centre of the circle within the triangular figure which represents one-eighth part of the whole unit cell.

We have described the semi-graphical method of analysis of the problem in detail because it permits easy visualisation of the procedure. But the results of the semi-graphical analysis may be prone to inaccuracy because of the minor graphical errors and it is especially complicated to do for $R>2$ which involves larger numbers of points and therefore the analysis of more arcs. Besides we are concerned with the theoretical aspects of this problem and hence it is preferable to develop the analytical expressions for defining the small area components which comprise the one half central square.

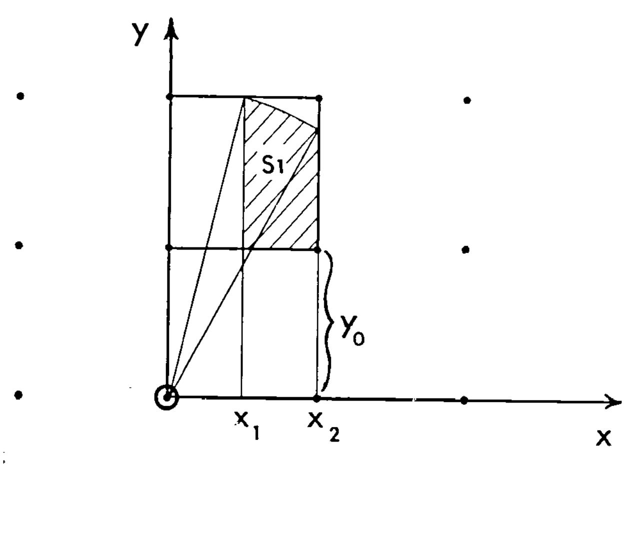

Figure 4.

In addition to the two expressions given in equation (13) which define the points of intersection of any arc with the two axes bounding the central square we may also calculate the point at which each arc intersects the diagonal of the square. Thus

$$\triangle_1 = \triangle_2 = \frac{(\nu+\eta) \pm \sqrt{2R^2 - (\nu-\eta)^2}}{2} \quad \ldots\ldots\ldots\ldots(14)$$

We may also find the points of intersection of pairs of arcs with one another. In order to do this we have used yet another form of coordinate notation to define the (x, y) system of Cartesian coordinates whose axes are parallel to the axes of the (ν, η) system. Then the coordinates of the point of intersection of any two arcs may be determined from the equations:

$$\begin{aligned} x^2 - \delta x . x + \left(\frac{\delta x^2 + \delta y^2}{4} - R^2 \frac{\delta y^2}{\delta x^2 + \delta y^2}\right) &= 0 \\ y^2 - \delta y . y + \left(\frac{\delta x^2 + \delta y^2}{4} - R^2 \frac{\delta x^2}{\delta x^2 + \delta y^2}\right) &= 0 \end{aligned} \quad \ldots\ldots\ldots\ldots(15)$$

where δx and δy are the coordinates of the second point from which an arc has been drawn with respect to the coordinates of the first point (x, y).

Because of the coincidence of the two systems of coordinates it is easy to convert from the (x, y) into $(\triangle_1, \triangle_2)$. by

$$\left.\begin{aligned}\triangle_1 &= x - \nu_0 \\ \triangle_2 &= y - \eta_0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(16)$$

The area of each part of the figure which is bounded on one side by an arc can now be found from the expression:

$$S_1 = \int_{x_1}^{x_2} (R^2 - x^2)^{\frac{1}{2}} . dx - (x_2 - x_1) y_0 \quad \ldots\ldots\ldots\ldots(17)$$

The meaning of the various terms in this equation is defined in *Figure* 4. After some transformation of this equation, and introducing the convention that $\frac{x}{R} = \sin \theta$, we obtain the final equation for calculating the areas in a useable form

$$S_1 = \frac{R^2}{2}[\cos(\theta_1 + \theta_2) . \sin(\theta_2 - \theta_1) + (\theta_2 - \theta_1)] - (x_2 - x_1) y_0 \quad \ldots\ldots\ldots\ldots(18)$$

If we denote by n that number of points covering the circle as the centre of it is moved into different positions of the central square and if we denote by s_i the area of each small figure as a fraction of the total area of the central square, we may obtain the mathematical expectation, $E(n)$ of the number of points which might be counted

$$E(n) = \Sigma n_i . s_i \quad \ldots\ldots\ldots\ldots(19)$$

If we determine the deviation, v_i, of an individual count from the expectation

$$v_i = n_i - E(n)$$

then we may obtain an expression for the mean square error of measurement of the area of a circle by a square point pattern as

$$m = \sqrt{\Sigma v^2 . s} \qquad \text{...........(20)}$$

The relative error, f, may now be expressed by

$$f = \frac{m}{E(n)} \qquad \text{...........(21)}$$

and we will normally express this as a percentage value.

We have tested the behaviour of a series of circles with radii, R=2, 3, 4 and 5 units, employing both the semi-graphical and analytical approaches outlined above. The results of these tests are given in Table 1.

TABLE 1

Summary of the results of the analysis of the displacement of circles with different radii with respect to a square point pattern.

R=2		R=3		R=4		R=5	
n	s	n	s	n	s	n	s
10	0.010	26	0.056	46	0.001	75	0.034
11	0.178	27	0.165	47	0.035	76	0.032
12	0.220	28	0.372	48	0.061	77	0.129
13	0.417	29	0.318	49	0.136	78	0.261
14	0.175	30	0.061	50	0.301	79	0.311
		31	0.008	51	0.281	80	0.176
		32	0.020	52	0.185	81	0.057
$S=E(n)$	12.57	28.27		50.27		78.54	
m	1.01	1.16		1.34		1.34	
f	8.0%	4.1%		2.7%		1.7%	

We see from Table 1 that as the radius of the circle is increased and therefore as n increases, the distribution of n, and therefore the deviations from $E(n)$ trend towards a normal distribution. For R=5 we have a distribution in which the skewness is only −0.456 and the excess is 0.180 which indicates a fairly good approximation to the normal distribution.

We turn now to the study of the influence of rotation of the overlay with respect to the parcel being measured. We must introduce two additional concepts in order to make this analysis. In the first place we must revise our definition of the unit cell to mean the square figure with side length d_2 which has a particular point on the overlay situated at the centre of it. We assume that such a cell is located at the edge of the parcel and may be intersected in any position and in any direction by its perimeter. We suppose, moreover, that the size of the unit cell is small enough for us to assume that the portion of the perimeter within the unit cell may be regarded as a straight line. Then the unit cell will be truncated by a straight line and will be divided into one part which is greater than one half of its area and another part which is less than one half of the area.

Although an arbitrary straight line can intersect the unit

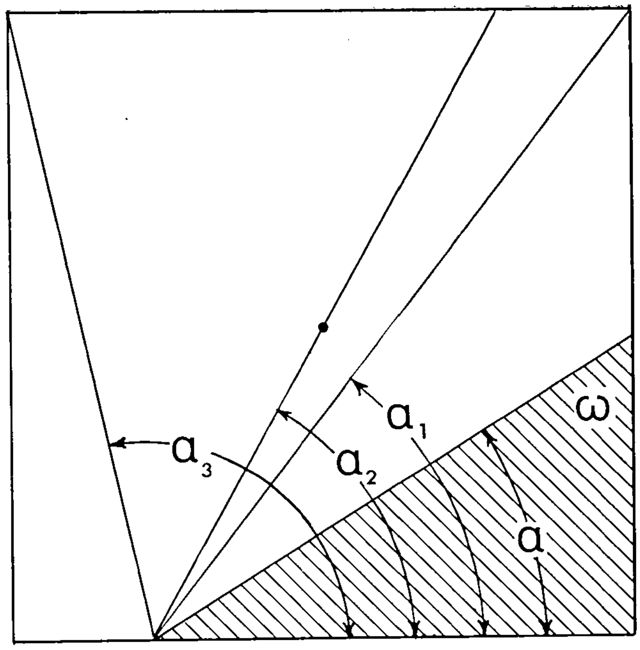

Figure 5.

cell in any position and at any angle, we can, by the symmetry of the figure, confine our attention to only one side of the cell, as indicated in *Figure* 5. The coordinate, x, of the point of intersection along the abscissa will have equal probability within the limits from $x=0$ to $x=1$; the angle α will also have equal probability within the limits $\alpha=0$ to $\alpha=\pi$. For some arbitrary point with abscissa x, the angle α can occupy one of three positions which we can define immediately. These are

$$\left.\begin{aligned} \alpha_1 &= \cot^{-1}(1-x) \\ \alpha_2 &= \cot^{-1}(1-2x) \\ \alpha_3 &= \pi - \cot^{-1}x \end{aligned}\right\} \qquad \text{...........(22)}$$

These angles are illustrated in *Figure* 5.

If we denote by ω the area which is cut off from the unit cell by the straight line, the value of it may be found from a series of equations, which depend upon the size of α.

Thus, if the line makes an angle which lies within the limits $0-\alpha_1$, then

$$\omega_1 = \tfrac{1}{2}(1-x)^2 \tan\alpha$$

If the angle α lies between the limits $\alpha_1-\alpha_2$, then

$$\omega_2 = (1-x) - \tfrac{1}{2}\cot\alpha$$

If the angle lies between the limits $\alpha_2-\alpha_3$, then

$$\omega_3 = x + \tfrac{1}{2}\cot\alpha$$

If the angle α lies between the limits α_3 and π, then

$$\omega_4 = \frac{-x^2}{2}\tan\alpha$$

The mean square value of the area $\overset{2}{\bar{\omega}}$ for a constant value of x may be obtained from the formula

$$\overline{\omega}^2=\frac{1}{\pi}\left[\int_0^{\alpha_1}\omega_1^2.d\alpha+\int_{\alpha_1}^{\alpha_2}\omega_2^2.d\alpha+\int_{\alpha_2}^{\alpha_3}\omega_3^2.d\alpha+\int_{\alpha_3}^{\pi}\omega_4^2.d\alpha\right] \quad \ldots\ldots\ldots\ldots(23)$$

Substituting in equation (23) the different values for ω_i given above and integrating this expression we finally arrive at a solution.

$$\overline{\omega}^2=\frac{1}{4\pi}[2-\pi-3x+3x^2+4\pi x^2]+$$
$$+\frac{1}{4\pi}\begin{Bmatrix}[1-4(1-x)^2-(1-x)^4]\text{ arc cot }(1-x)+\\ (1-4x^2-x^4)\text{ arc cot }x\end{Bmatrix}+$$
$$+\frac{1}{\pi}(1-2x)\text{ arc cot }(1-2x)+$$
$$+\frac{1}{\pi}\begin{Bmatrix}-\tfrac{1}{2}(1-x)\log_e[1+(1-x)^2]+\tfrac{1}{2}\log_e[1+(1-2x)^2]\\ -\frac{x}{2}\log_e(1+x^2)\end{Bmatrix} \quad \ldots\ldots\ldots\ldots(24)$$

The minimum value of $\overline{\omega}^2$ will correspond to the values of $x=0$ and $x=1$. In both of these cases

$$\overline{\omega}^2=\frac{1}{4\pi}(2-\pi/2)$$

or

$$\overline{\omega}=0.1848$$

The maximum value of $\overline{\omega}$ will occur when $x=0.5$. In this case,

$$\overline{\omega}^2=\frac{1}{4\pi}\left(\frac{5}{4}-\frac{1}{8}\text{ arc cot }\tfrac{1}{2}-2\log_e\frac{5}{4}\right)$$
$$=0.2436$$

We must now find the value of $\overline{\omega}$ for variation in x between 0 and 1. Denoting this by Ω we will have

$$\Omega^2=\int_0^1\overline{\omega}^2.dx$$

Substituting here the value of $\overline{\omega}^2$ from (24) and after some rearrangement of the expression we finally obtain

$$\Omega^2=\frac{1}{60\pi}(\pi+4+2\log_e 2) \quad \ldots\ldots\ldots\ldots(25)$$
$$\Omega=0.2127$$

We must now determine the mean square value of the length of the straight line element, l, enclosed within the unit cell which we assume represents part of the perimeter of the parcel. This is obviously dependent upon the values of x and α and, as before, the expression will vary according to the size of α. Thus, if α lies between 0 and α_1 we use the expression

$$l_1=(1-x)\sec\alpha$$

If α lies between α_1 and α_3, then

$$l_2=\operatorname{cosec}\alpha$$

and if α lies between α_3 and π,

$$l_3=x.\sec\alpha$$

The mean square value for l for a constant value of x may now be defined as:

$$\overline{l}^2=\frac{1}{\pi}\left[\int_0^{\alpha_1}l_1^2.d\alpha+\int_{\alpha_1}^{\alpha_3}l_2^2.d\alpha+\int_{\alpha_3}^{\pi}l_3^2.d\alpha\right] \quad \ldots\ldots\ldots\ldots(26)$$

If we substitute here for l_i, integrate the resulting expression and then simplify the result we obtain

$$\overline{l}^2=\frac{2}{\pi}$$

or

$$\overline{l}=0.7979 \quad \ldots\ldots\ldots\ldots(27)$$

We can see that l varies within the limits $0\leqslant l\leqslant\sqrt{2}$ and that the mean square value $\overline{l}^2$ is wholly independent of x. Thus a displacement of the pattern of points has no influence upon the mean square value of an arbitrary straight line which intersects the pattern at any arbitrary angle.

We have already assumed that the straight line is the representation of the parcel perimeter within any given cell. Thus we substitute for the true perimeter of the parcel a series of short rectilinear elements, one lying in each cell

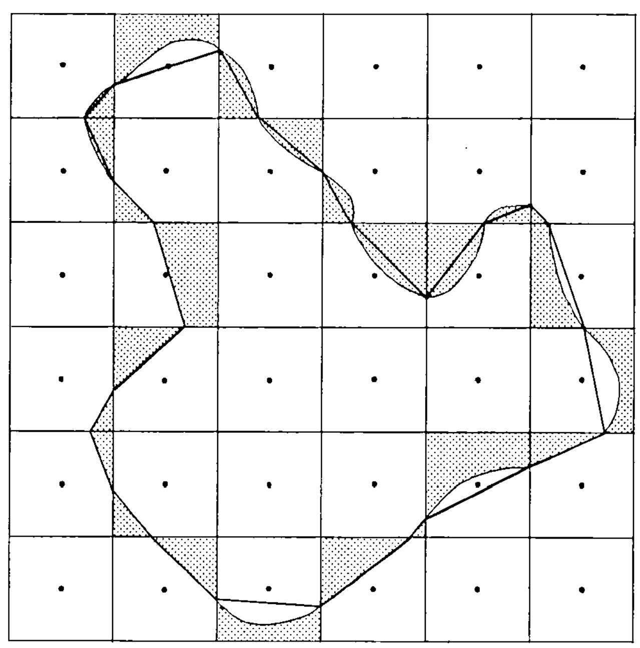

Figure 6.

of the point pattern. The smaller is the size of the unit cell compared with the area of the parcel, the less will be the relative error in measurement which arises from this assumption.

In each of the cells which coincide with the parcel perimeter there will occur either a positive or negative error of measurement according to the position where the rectilinear element intersects the unit cell. As may be seen in *Figure* 6, the shaded part of each cell which lies outside the perimeter of the parcel will give rise to positive errors in measurement; the shaded parts which lie inside the perimeter of the parcel will represent negative errors. The maximum possible error will clearly be equal to $\pm 0.5\ s$ (where s is the area of the unit cell). The mean square error will equal Ω which has already been defined by equation (25) as having the constant value of 0.2127. For uniformity of symbolisation in the final expressions, we put $\Omega = k_1 = 0.2127$.

The standard error of the whole parcel will be

$$m = k_1 \sqrt{n_p} \qquad \text{...........(28)}$$

where n_p is the total number of cells which intersect the perimeter of the parcel. This quantity may be determined from

$$n_p = k_2 . L \qquad \text{...........(29)}$$

where L is the cumulative length of all the small rectilinear elements l_i which we have substituted for the true perimeter of the parcel. The coefficient k_2 may be expressed as

$$k_2 = \frac{1}{l} \qquad \text{...........(30)}$$

which, from (27) is constantly equal to 1.2533.

The true perimeter of the parcel may be related to the area of the figure enclosed by all the straight line elements by the expression

$$L = k_3 \sqrt{S} \qquad \text{...........(31)}$$

and the coefficient k_3 depends upon the shape of the parcel. We have evaluated this coefficient for the series of simple geometrical figures illustrated in *Figure* 7 and the resulting values for k_3 are tabulated below (Table 2).

If we now combine the three expressions (29), (30) and (31) we may rewrite equation (28) in the form

$$m = k_1 (k_2 . k_3)^{\frac{1}{2}} . \sqrt[4]{S} \qquad \text{...........(32)}$$

so that the standard error of area measurement by counting points is proportional to the fourth root of the area of the parcel. This conclusion is to be compared with the relationship of the standard error of measurement of area by planimeter which many writers[8, 9, 10] have shown to be proportional to the square root of the area. Köppke[5] has argued that the mean square error of area measurement by counting points is also proportional to the square root of the area but we cannot agree with this conclusion.

In order to make practical use of equation (32) and, in particular, to evaluate the density of points needed to measure area to a given standard of accuracy, we will make

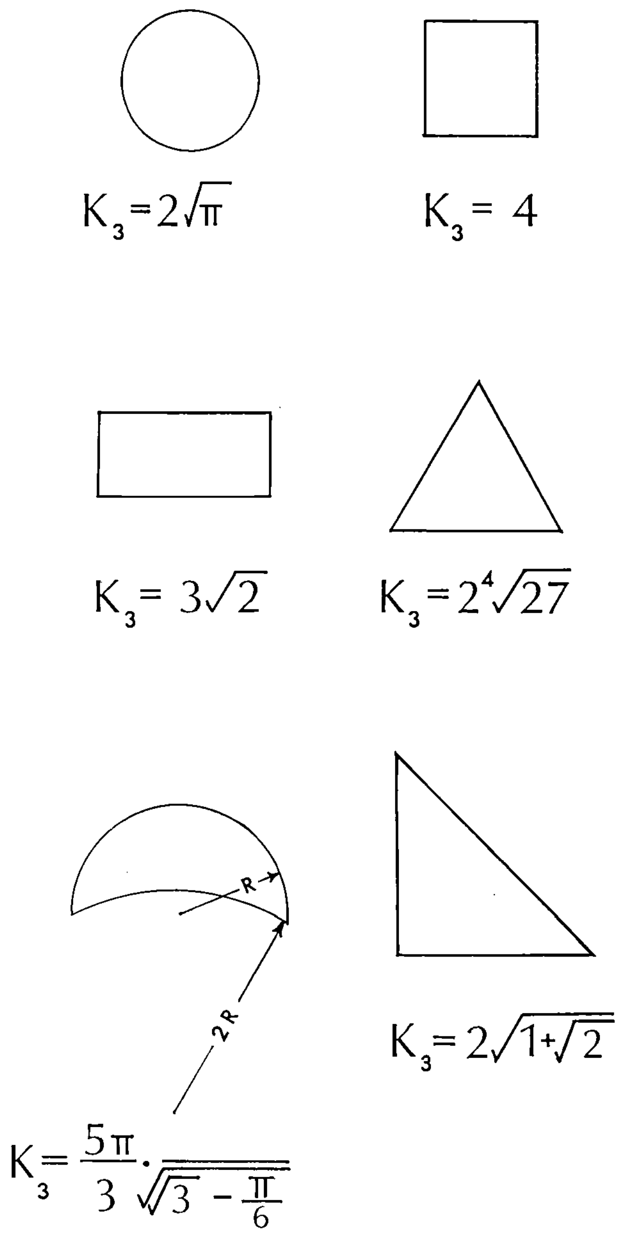

Figure 7.

use of the relative value of the mean square error. Thus we may rewrite (32) in the form

$$f = \frac{m}{S} = k_1 (k_2 . k_3)^{\frac{1}{2}} . S^{-\frac{3}{4}} \qquad \text{...........(33)}$$

If we convert this to logarithms, then we may combine the three coefficients as a single term

$$K = 2 + \log k_1 + \tfrac{1}{2} \log k_2 + \tfrac{1}{2} \log k_3 \qquad \text{...........(34)}$$

and the relative error may be expressed as a percentage in the form

$$\log f\% = K - 0.75 \log S \qquad \text{...........(35)}$$

The value of the coefficient K will vary only with variations in k_3 which we have seen represents some measure of parcel shape. For the series of simple geometrical figures illustrated in *Figure* 7 the values for k_3 and hence of K will be as follows:

C

TABLE 2

The influence of parcel shape upon the coefficients k_3 and K for simple geometrical figures.

Figure	k_3	K
Circle	3.5449	1.652
Square	4.0000	1.678
Rectangle with ratio of sides 1:2	4.2427	1.691
Equilateral triangle	4.5586	1.706
Lune formed by two circular arcs with ratio of radii 1:2	4.4730	1.716
Isosceles right-angled triangle	4.8284	1.719

From this table we can see that the value of K varies only slowly with shape. We have calculated equation (35) for different values of S and for the two limiting values of K, namely those for the circle and for the right-angled isosceles triangle. These are tabulated in Table 3. Note that the assumption that the parcel perimeter has been replaced by a series of short rectilinear elements will apply only to the circle and to the lune in this particular series of geometrical figures.

TABLE 3

Values for relative error of measurement (f%) for different sizes of parcel, S, employing the values of K determined for the circle (f_1) and for the isosceles right-angled triangle (f_2).

S	f_1	f_2	S	f_1	f_2
5	13.42%	15.66%	55	2.23%	2.60%
10	7.98	9.31	60	2.08	2.43
15	5.89	6.87	65	1.96	2.29
20	4.74	5.53	70	1.86	2.17
25	4.01	4.68	75	1.76	2.05
30	3.50	4.08	80	1.68	1.96
35	3.12	3.64	85	1.60	1.87
40	2.82	3.29	90	1.54	1.80
45	2.58	3.01	95	1.47	1.72
50	2.39	2.79	100	1.42	1.66

These values are also plotted in *Figure* 8

Clearly the equations (33) and (35), Table 3 and *Figure* 8 all demonstrate the well known fact that for a given error of measurement, e, the relative error, f, for a small parcel will be greater than the corresponding value for f for a large parcel because e represents a proportionately larger fraction of the smaller parcel. However it can also be seen from *Figure* 8 that as area increases the curves converge. That is to say, the differences between the relative errors of measurement for differently shaped parcels become smaller as the size of the parcels is increased. For parcels in which the count approaches 100 the difference between the relative errors of the two shapes of parcel studied above becomes less than 0.24%.

In order to test this theory for more elongated, less compact, parcels the values for f% have been calculated for a series of rectangles possessing the shorter pair of sides equal to unity and the longer pair of sides equal to 2, 5, 10,

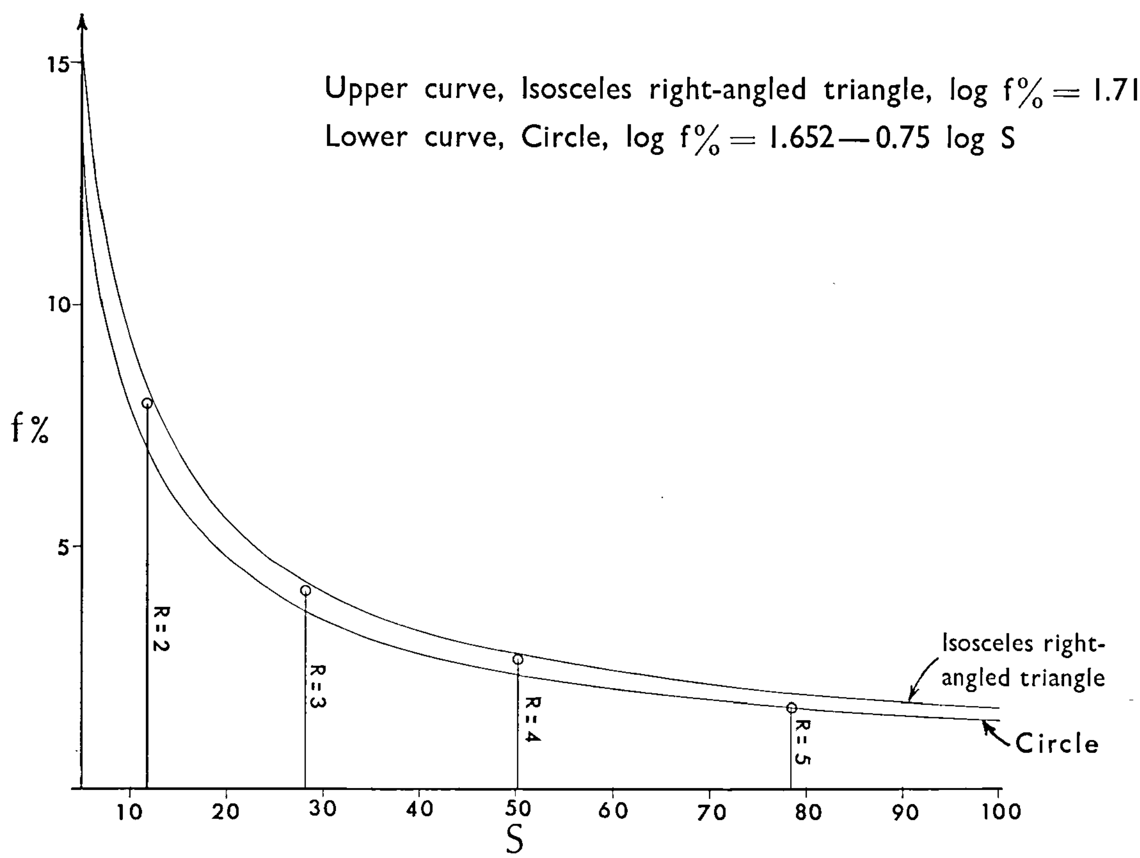

Figure 8.

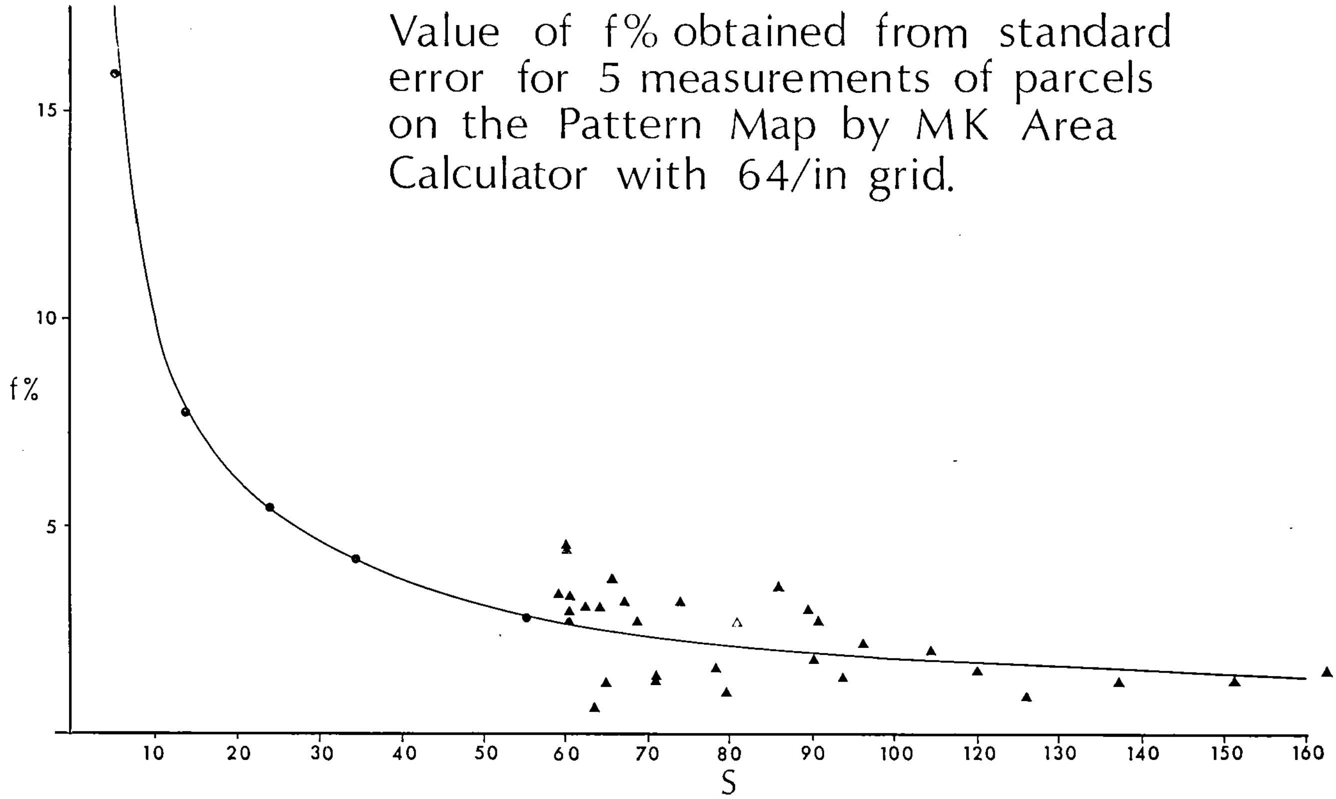

Figure 9.

25, 75, 100, 150 and 200 units. The extremely elongated rectangles might be interpreted as corresponding to the sort of linear features, such as rivers, roads or railways the areas of which might be required for land use or land classification purposes. The results of these calculations are presented in Table 4.

TABLE 4

Values for the relative error ($f\%$) of measurement of the areas of rectangles with different ratios between the lengths of the sides.

Ratio	1:2	1:5	1:10	1:25	1:75	1:100	1:150	1:200
Area (S)	$f\%$	$f\%$	$f\%$	$f\%$	$f\%$	$f\%$	$f\%$	$f\%$
5	14.7	16.5	18.8	23.0	29.8	32.0	35.4	38.0
10	8.7	9.8	11.2	13.7	17.7	19.0	21.0	22.6
25	4.4	4.9	5.6	6.9	8.9	9.6	10.6	11.4
50	2.6	2.9	3.3	4.1	5.3	5.7	6.3	6.7
75	1.9	2.2	2.5	3.0	3.9	4.2	4.6	5.0
100	1.5	1.7	2.0	2.4	3.2	3.4	3.7	4.0

Comparison of these figures with the corresponding values of Table 3 indicates that the rectangle with side lengths 5 units $\times$ 1 unit gives rise to a curve which lies close to that for the right-angled isosceles triangle. Not surprisingly the more elongated figures indicate larger relative errors but the important conclusion is that the curves still converge to an overall range of 2.5% for S equals 100.

It is also interesting to compare the values obtained from equation (35) for the four circles with different radii which were studied earlier (see Table 1) with reference to the frequency of different possible counts. The comparison between the two sets of relative errors is given in Table 5.

TABLE 5

The comparison between the relative errors ($f\%$) obtained from the analysis of the coincidence of overlay points (f_0) for circles of different radii and the relative errors of area measurement calculated from equation (35) (f_t).

R	S	m	f_0	f_t	f_0-f_t
2	12.57	1.01	8.0%	6.7%	+1.3%
3	28.87	1.16	4.1	3.7	+0.4
4	50.27	1.34	2.7	2.4	+0.3
5	78.54	1.34	1.7	1.7	0.0

The expression which will best fit the values of f_0 is

$$\log f\% = 1.652 - 0.75 \log S$$

The points corresponding to f_0 for the mean values for n derived from Table 1 are also indicated in *Figure* 8.

The discrepancy between f_0 and f_t recorded in the last column of Table 4 is interpreted by us as a small systematic error which results from our assumption that the perimeter in each cell is represented by a straight line, whereas this is demonstrably a false assumption when we measure the areas of circles by this method. However we may see in Table 5 and *Figure* 8 that the systematic error decreases rapidly as the area of the parcel increases and, in practice, if S is represented by a count of more than 50 points it can be ignored.

Experimental Tests

It is now desirable to see if the errors obtained in practical measurement correspond in size to those derived theoretically.

We have therefore analysed the results of 3,340 measurements made on the Pattern Map devised and constructed by C. J. McKay[11] specifically for the purpose of evaluating different instruments and techniques for area measurement.

The parcels on the Pattern Map have been constructed geometrically to provide an independent measure of their areas and most of them are originally based upon simple figures with rectilinear sides. Although groups of these simple figures have been combined in order to make parcels of irregular shape, the rectilinear outline has been retained. Thus measurements made on the Pattern Map do not represent a test for any errors arising from our assumptions about the shape of the perimeter within each unit cell.

The greatest frequency of parcels on the Pattern Map is for the smallest size class (0-0.999 cm^2) and the distribution of parcels by size yields a characteristically L-shaped pattern. This corresponds to the size distribution which is often encountered on maps and plans, certainly of Western Europe, but it is inconvenient in a study of the relative errors of measurement. In order to create some balance between the paucity of measurements for larger parcels and the considerable amount of data available for small parcels, we have selected a random sample from the measurements available for the smaller sizes classes and we have used all the measurements for parcels larger than 3.999 cm^2.

The measurements were made with the M.K. Area Calculator using the $\frac{1}{64}$ inch grid. Each parcel was measured by five random applications of the overlay. The sampling parameter which most closely corresponds to the theoretical mean square error, m_f, is the standard error of the single observation. This is expressed in units of the number of points counted in the column headed *m* in Table 6.

The tests made with the M.K. Area Calculator have already been described[2] where the unreliability of this instrument as a counting device has been emphasised. The demonstration that measurements made with the M.K. Area Calculator are commonly prone to negative gross errors was largely done by making a comparison of the measurements made in the bigger units, or blocks, on the Pattern Map. These blocks vary in size between approximately 100 cm^2 and 300 cm^2, corresponding to counts within the range 1,000-3,500 points. The difference between the measurements of each whole block with an area derived from the sum of the parcels contained within that block demonstrates that gross errors have often occurred in making meas ements with this order of magnitude. Although gross errors certainly occurred in measuring the areas of individual parcels these are much less common and the doubtful parcels have been eliminated from the present analysis.

TABLE 6

Relative errors of measurement of area on the Pattern Map obtained with M.K. Area Calculator.

Mean area S (count)	Class range (cm^2)	m (count)	f_0 (%)	f_t (%)	Difference (%)	Size of Sample
5.08	0-0.999	0.943	15.9	13.9	+2.0	30
14.31	1-1.999	1.205	7.7	7.1	+0.6	30
24.57	2-2.999	1.476	5.5	4.7	+0.5	15
34.79	3-3.999	1.541	4.2	3.7	+0.5	15

The values for f obtained for all parcels greater then 3.999 cm^2 are shown individually in *Figure* 9. The curve in *Figure* 9 is derived from the mean values of f given in Table 6 of the samples taken from the four smallest size classes and corresponds to the expression

$$\log f(\%) = 1.771 - 0.75 \log S$$

Table 7 provides some indication of the relative errors which are likely to arise in the measurement of extremely elongated parcels, exemplified on the Pattern Map by nine 'roads' which separate the ten blocks. These roads have been regarded as corresponding to extremely elongated rectangles for the purpose of determining theoretical values for f.

TABLE 7

Relative errors of measurement of extremely elongated parcels exemplified by the roads on the Pattern Map. Measurements made by M.K. Area Calculator.

Serial letter of road	Area (count)	Corresponding ratio of sides for rectangle	m (count)	f_0 (%)	f_t (%)	Difference (%)
m	11.43	1:108	1.643	14.4	17.5	−3.1
l	12.21	1:121	1.923	15.7	17.2	−1.5
r	15.88	1:158	2.074	13.1	15.1	−2.0
s	17.92	1:170	0.894	5.0	14.0	−9.0
t	22.46	1:215	3.493	15.5	12.5	+3.0
o	23.37	1:232	2.191	9.4	12.4	−3.0
p	23.95	1:238	2.074	8.7	12.2	−3.5

(Note that two roads, serials *n* and *q*, are missing from this table. These were too long to be measured with the M.K. grid as single parcels).

Both Tables 6 and 7 indicate that there is some measure of agreement between theory and practice. In both of the tables a positive difference indicates that the results obtained by the M.K. Area Calculator were less precise than the theory suggests; a negative difference indicates that the actual results are somewhat better than one might expect from our theoretical consideration of the problem. Table 7 indicates some fairly large fluctuations but this is to be expected from the very small sample which we have been able to study. The consistently positive difference in Table 6 really depends upon what numerical value is apportioned to the term K in equation (35) in order to calculate the theoretical value, f_t, corresponding to the mean area of each class. The figures tabulated in Table 6 were calculated from a mean value of $K=1.719$ derived from Table 2, but a more realistic value, $K=1.771$ has been found in fitting the curve shown in *Figure* 9 to the observed data.

Comparison of the Theoretical Accuracy of Counting with other Methods of Area Measurement

The analysis thus far has been made with reference to unit distance between points, d, and of unit area, s, of each square cell. Even the examination of the experimental data has been made with reference to the number of points counted rather than their metric equivalents derived from equation (1). Generally, however, the area of a parcel shown on a map will be required in units of measurement which have some practical significance. Moreover if we wish to compare these results with the equivalent studies made for other techniques of area measurement, we must use a common standard of measurement.

In order to express the relative error, f, with respect to area measured in square centimetres, equation (35) must be rewritten as

$$\log f(\%)=K+\frac{3}{2}\log t-0.75\log S' \qquad \text{...........(36)}$$

where t is the side length of the unit cell in centimetres and the area of the parcel, S', is expressed in square centimetres. *Figure* 10 illustrates the relationship between f and S' for a series of different sizes of square point pattern. These values have been calculated from equation (36) using for K the mean value of 1.719.

Most of the work on the accuracy of area measurement concerns the evaluation of different kinds of planimeter. As an example of the kind of precision found in the three basic types of instrument we have reproduced, in *Figure* 11, the three curves deduced by Zill[9] for the ordinary compensating polar planimeter, for the polar disc planimeter and for the rolling disc planimeter. We compare these with the corresponding curves deduced from equation (36) for the theoretical precision of square grids where $t=0.2$, 0.4, 0.6 and 0.8 cm. This indicates that the precision of point counting with a square grid for $t=0.2$ cm compares well with Zill's curve for the polar planimeter but no counting method can compare in precision with the more sophisticated polar disc and rolling disc instruments.

Comparative Speeds of Measurement

We must now investigate the time which is required in order to make area measurements by the different methods and compare this with the sort of precision of measurement which may be attained. It has already been noted that tests

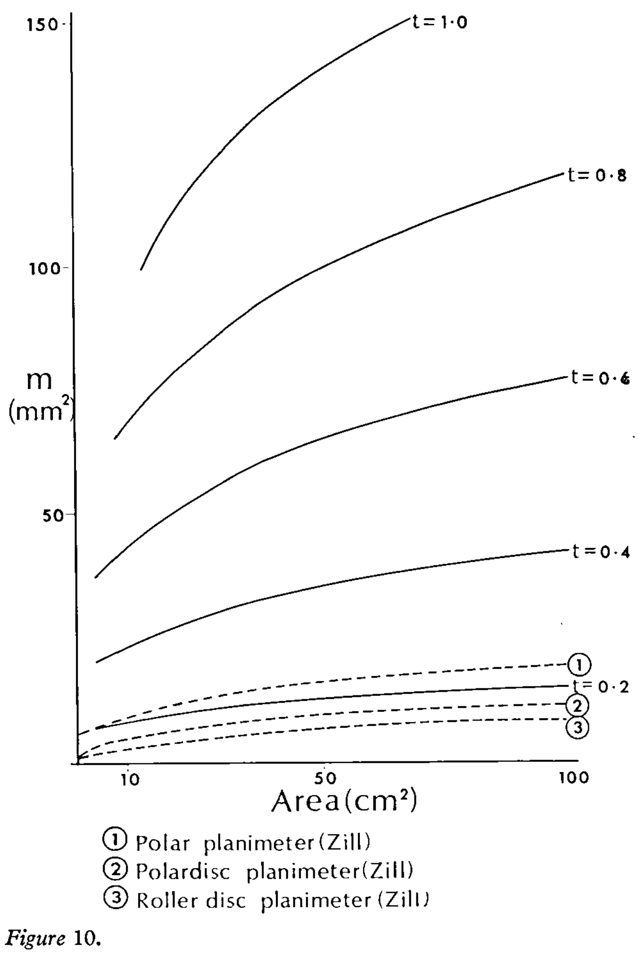

Figure 10.

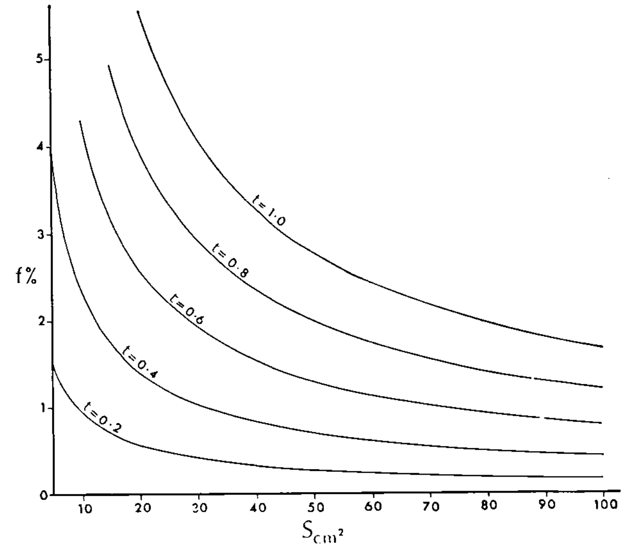

Figure 11.

of the different techniques and instruments suggest an average counting speed of about 1.5 points per second. This includes the time needed to read and book the results at the approximate rate of one entry *per* count of 25 points. Admittedly this rate can be improved upon in making measurements of small parcels because the hand which marks the points counted does not have to be moved from place to place, but against this must be offset the decrease in rate which results when a large number of points have to be counted. Although we appreciate that the counting rate may vary significantly between individual operators, we believe that any prolonged series of measurements would yield an average counting speed not much different from the 1.5 points per second which we have adopted.

A measurement by planimeter depends upon making two readings of the vernier each time the instrument is traced round a parcel. This, combined with booking the two measurements comprises about 15 seconds and this is more or less constant for any size of parcel. The actual tracing time depends, again, upon the skill of the operator and also upon the complexity of the outline, but Baer's estimate of 0.2 cm per second[12] seems to be realistic. For planimeter measurements we have, therefore, assumed square parcels of different area to determine the length of perimeter and we have used the two criteria to estimate the measurement time.

In *Figure* 12 we have obtained seven curves which equate precision, m, with time, T, for Zill's three planimeter equations and four different sizes of point overlay. The broken lines on this figure locate parcel sizes of 20, 50 and 100 cm² for the counting measurements and of sizes 5, 20, 50, 100, 200 and 300 cm² for planimeter measurements.

We can see, immediately, the difference in character of these sets of broken lines. Those calculated for the three planimeters are vertical, indicating that the execution time is the same for each kind of instrument. The broken lines linking the curves based on counting illustrates that as the separation between points is reduced so the number to be counted rises in geometrical progression, and thus the time needed to make the measurements increases in a similar fashion.

The concentration of all the planimeter curves close to the origin of the axes indicates quite clearly the superiority of these instruments both with respect to precision and to

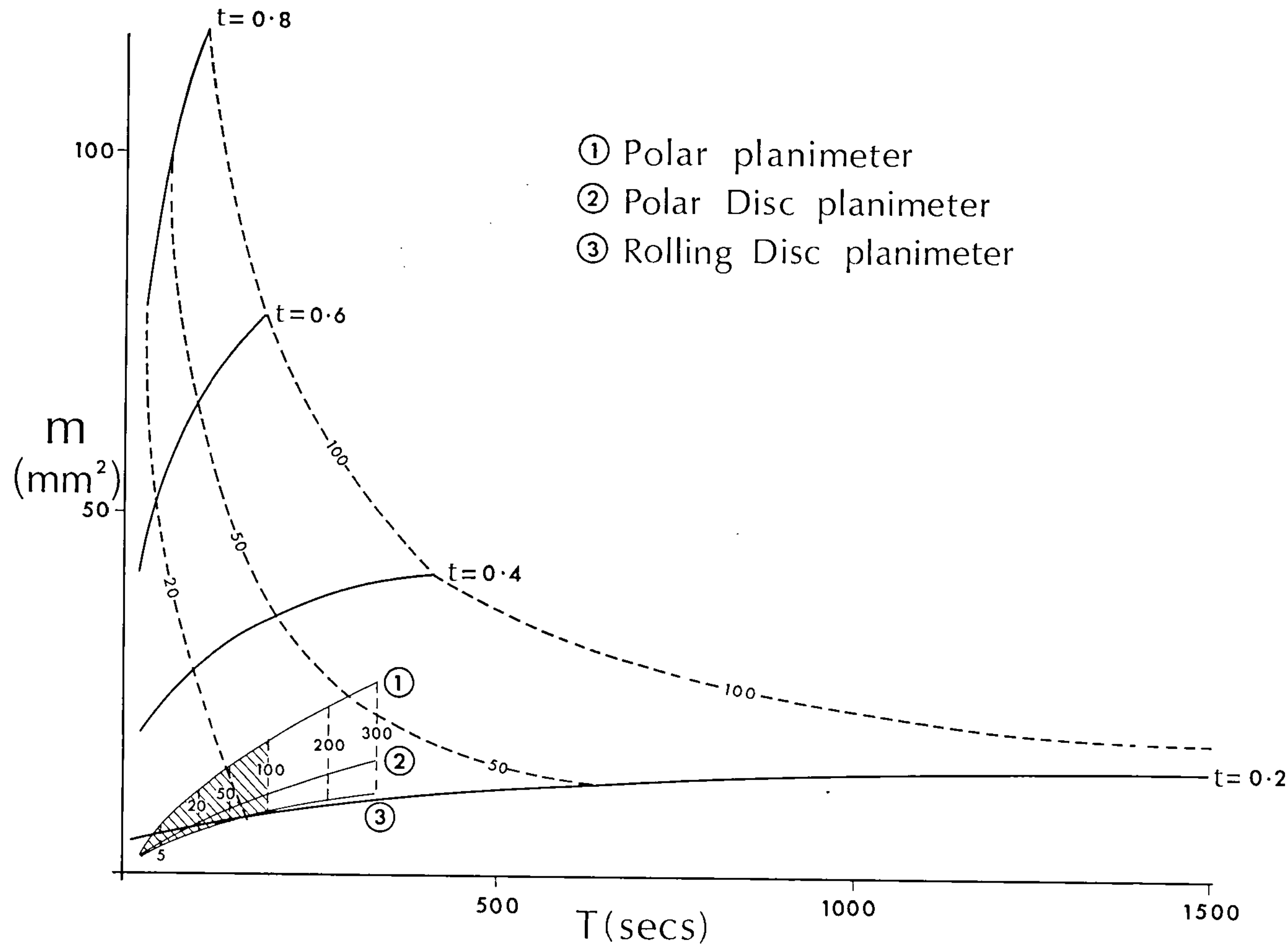

Figure 12.

economy of measurement time. It is only the small parcels with area less than about 3 cm² which can be measured more quickly and with equivalent precision by counting with a 0.2 cm or finer grid; all larger parcels are more efficiently measured by planimeter.

Conclusions

If this is so, what justification is there in using point counting techniques in preference to planimeter measurements?

There are, we believe, two reasons why area measurement by these methods can still be regarded as economic and adequate. First we must bear in mind that the measure of precision, characterised by the use of the mean square error, m, in square millimetres is unrelated to parcel size. We emphasise the importance of using the relative error in any consideration of area measurement because it is much more useful from a practical point of view. For most purposes it is sufficient to demonstrate that the relative error of a series of measurements does not exceed some criterion such as 1% or 1.5%. We have already shown that this order of accuracy is related to the number of points counted. For example, if we specify that f should not exceed 1.5% then the minimum count of points in a parcel should be about 100. Since this is unrelated to the separation of the points on the overlay we suggest that we can discard the conventional notion of using the same overlay for all parcels, irrespective of their size, but we should employ several. We might use the following limiting values of t for different sizes of parcel:

TABLE 8

	S	t
Less than	4 cm²	0.2 cm
	16	0.4
	36	0.6
	64	0.8
Greater than	100	1.0

The individual point patterns might be reproduced on different sheets of plastic, or they might be superimposed one upon the other using different colours to separate one from another and avoid confusion in counting. Then the operator makes a rough guess at the area of a parcel and then selects the most suitable point separation.

Since no count should differ from 100, we can estimate that the single measurement of any parcel will be slightly more than one minute. The second advantage possessed by point counting is that this is a more convenient technique for measuring the areas of separate parcels when we only need to know their total area. This may be exemplified by the requirement to measure the total area of woodland shown on a map and is clearly a common problem in many kinds of land use and land classification studies. The counting procedure is organised so that each parcel to be measured is counted during each application of the overlay. Corresponding measurements by planimeter would entail setting up the instrument separately for each parcel in turn.

In this context we should note that an increase in the number of parcels measured will result in some loss of accuracy. An n-fold increase in the number of parcels measured results in an increase in the relative error which is proportional to $\sqrt[4]{n}$. Thus, if we denote the relative error in measuring and area which is contained in n separate parcels by f_n,

$$\log f_n(\%) = \log f(\%) + \frac{1}{4} \log n \quad \text{...........(37)}$$

The errors which will arise in this procedure approximate to the binomial distribution. Thus we may also determine the relative error from the expression

$$f = \left(\frac{1-p}{pn}\right)^{\frac{1}{2}} \quad \text{...........(38)}$$

where n is the number of points which coincide with the parcels to be measured and p is the probability that any point lies within these parcels. This is equivalent to the ratio of the sum of the areas of the parcels measured to the area of the whole map or region.

The relative errors calculated from equation (38) will be greater than are the corresponding errors of measurement of the single parcel having the same area. This is accounted for by the increase in the total length of the perimeters of n parcels compared with the length of the perimeter of a single parcel having the same area.

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Metrication of Ordnance Survey Maps

The first Ordnance Survey maps to be based completely on metric measurements will be published in the autumn of this year. The changeover will be gradual and at this stage will be limited to the large-scale O.S. maps, including the six-inch-to-the-mile series. Metrication of the popular one-inch and smaller scale maps is still under study.

For many years the Ordnance Survey has used a metric grid on its maps; the sizes of the map sheets themselves have been based on this grid and correspond to metric dimensions on the ground. The scales of most of the maps series are already in decimal form, *e.g.* 1:2500, 1:25 000); the only exceptions being the one-inch and the six-inch maps.

The changes being introduced are as follows:—

1:1250 and 1:2500 Scales

On new and revised sheets and heights of bench marks will be shown to two decimal places of a metre and of spot heights to one decimal place. The mereings of administrative boundaries will be shown in metres to two decimal places. On the 1:2500 maps areas of parcels of land will be given in hectares to three decimal places and also in acres as hitherto. The first metric maps at these scales will appear in October 1969 but it will be many years before all 1:1250 and 1:2500 maps (there are some 150 000 of them) are converted to metric form.

Six-inch and 1:25 000 Scales

The six-inch (1:10 560) scale will be replaced by the 1:10 000 with metric contours. The contour interval will be 10 metres in the more mountainous areas and 5 metres in the remainder of the country. The first sheets at the 1:10 000 scale will be published in December 1969, but it will be many years before the country is covered with a homogeneous series at this scale. When, in about 1985, the present Provisional series has been replaced over the whole country by Regular sheets at the 1:10 560 or 1:10 000 scale, the map will still exist in three forms:—

(1) 1:10 000 scale with metric contours at 5 metre or 10 metre interval. (68 *per cent*).
(2) 1:10 000 scale with 25 feet contours labelled with the equivalent metric values. (22 *per cent*).
(3) 1:10 560 with 25 feet contours. (10 *per cent*).

These will eventually be converted to 1:10 000, but in order to minimise inconvenience while this situation continues the Ordnance Survey will supply reductions of the 1:10 000 map to 1:10 560 and enlargements of the 1:10 560 to 1:10 000 for limited areas on request. These reductions or enlargements will be in a single colour.

Publication of the 1:25 000 map with metric contours will follow the 1:10 000. The contour interval will be consistent over the whole of the 1:25 000 sheet.

Bench Mark Lists

The heights of bench marks will be given in metres and in feet in bench mark lists.

Identification of Metric Sheets

All metric sheets will carry the British Standards Institution metric symbol and also a prominent marginal note 'Heights in Metres'.