The expected difference between $N(f)$ and $MF(f)$

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We all agree on the "wrong". This talk is about the "very badly".

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Kelly computes $N(f) = 0$ and $MF(f) = 2k$.

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When f is chosen at random, what is our statistical expectation for the quantity $MF(f) - N(f)$?

Randomness in free groups

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We can measure the size of a subset $S \subset G$ using the asymptotic density.

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This limit may not exist, so we also talk about these:

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These densities represent that probability that a random "very long" element of G is in S.

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So combinatorially, the set of endomorphisms $G \to G$ is G^n.

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So "almost all" maps have Nielsen number "arbitrarily large".

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Generally, if $f : G^n \to \mathbb{R}$ is any real valued function of homomorphisms, define the expectation of f as:

$$\mathcal{E}(f) = \lim_{p \to \infty} \frac{1}{\# G_p^n} \sum_{\varphi \in G_p^n} f(\varphi).$$

And similarly $\underline{\mathcal{E}}(f)$ and $\overline{\mathcal{E}}(f)$.

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The "best" result would be that $\mathcal{E}(MF - N) = 0$ and $\mathcal{E}(N/MF) = 1$.

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So we expect $\log \frac{N(f)}{MF(f)} = 0$.

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If that's too hard, we will try to show $\mathcal{E}(N/MF) = 1$.

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Let W_n be the set of φ with $MF(\varphi) = N(\varphi)$, so we want to compute $D(W_n)$.

It turns out that $D(W_n)$ depends on n (# of generators).

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The remnant is the middle subword in each position that doesn't cancel in any product.

Theorem (Wagner 1999)

If φ has nontrivial remnant in each generator, then it is the induced homomorphism for a map $f : X_n \to X_n$ with these properties:

- *f has a fixed point for each occurence of $a_i^{\pm 1}$ inside $\varphi(a_i)$*
- *The fixed point classes are determined by equalities among the "Wagner tails"*

For example:

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Also $MF(f) = 5$, since we have 5 fpcs of 1 point each.

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This is something we should be able to count combinatorially.

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Theorem (Brimley, Griisser, Miller, S. 2012)

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In the case $n = 2$, the answer is no.

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We show that $\underline{D}(T_3) > .00003$ which gives the bound above.

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For $n = 2$ the best bounds we have are:

$$.5 < \underline{D}(W_2) \leq \overline{D}(W_2) < .99997$$

Expectations

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The "best" result would be $\mathcal{E}_n(MF - N) = 0$ and $\mathcal{E}_n(N/MF) = 1$ for all n.

Expectations

Let's discuss $\mathcal{E}(MF - N)$ and $\mathcal{E}(N/MF)$.

These probably will depend on n, so let's write $\mathcal{E}_n(MF - N)$ and $\mathcal{E}_n(N/MF)$.

The "best" result would be $\mathcal{E}_n(MF - N) = 0$ and $\mathcal{E}_n(N/MF) = 1$ for all n.

But the fact that $D(W_2) < 1$ easily shows that $\mathcal{E}_2(MF - N) > 0$.

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Proof.

$$\underline{\mathcal{E}}_2(MF - N) = \liminf_{p\to\infty} \frac{1}{\# G_p^2} \sum_{\varphi \in G_p^2} MF(\varphi) - N(\varphi)$$

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$$\geq \liminf_{p\to\infty} \frac{1}{\# G_p^2} \sum_{\varphi \in W_2^c \cap G_p^2} 1$$

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\underline{\mathcal{E}}_2(MF - N) &= \liminf_{p\to\infty} \frac{1}{\# G_p^2} \sum_{\varphi \in G_p^2} MF(\varphi) - N(\varphi) \\
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&\geq \liminf_{p\to\infty} \frac{1}{\# G_p^2} \sum_{\varphi \in W_2^c \cap G_p^2} 1 = \liminf_{p\to\infty} \frac{1}{\# G_p^2} \#(W_2^c \cap G_p^2)
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$\square$

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We have tried many times to prove that these are finite, but the combinatorics is very difficult.

So we will instead look at $\mathcal{E}_n(N/MF)$. (We want this to be 1.)

Approach

Remember:

Lemma

If there are no equalities between the Wagner tails, then φ *is Wecken.*

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To measure $\mathcal{E}_n(N/MF)$, we need to consider when there are some equalities between the Wagner tails.

We need some condition which lets us predict when a Wagner tail is repeated

A necessary condition for repeated Wagner tails:

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Lemma

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$$\varphi(a) = waS, \quad \varphi(b) = wbT.$$

A necessary condition for repeated Wagner tails:

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A repeated Wagner tail must be completely outside of the remnant subword.

Proof.

If w is a repeated Wagner tail, this means that we have something like

$$\varphi(a) = waS, \quad \varphi(b) = wbT.$$

and since w will cancel in the product $\varphi(a^{-1})\varphi(b)$, it is outside of the remnant. □

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The right side above is the proportion of the WTs which appear inside the remnant.

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$$\varphi : \begin{array}{rcl} a & \mapsto & = b\underline{ba}c \\ b & \mapsto & = c\underline{cbca^{-1}}a^{-1} \\ c & \mapsto & = b\underline{acc}a^{-1} \end{array}$$

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Here we have 4 WTs inside the remnant, and 5 total WTs, so

$$\frac{N(f)}{MF(f)} \geq \frac{4}{5}$$

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To get

$$\frac{\#\text{WTs in remnant}}{\#\text{WTs}} \approx 1,$$

we want the remnant subwords to be as large as possible.

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$$|\operatorname{Rem}_\varphi a_i| > r|\varphi(a_i)|$$

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Theorem

(S. 2010) $D(R_r) = 1$ *for any* r.

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We should also expect proportion r of the WTs to be inside the remnant, provided that the WTs are "uniformly distributed" in the image words.

We need to make sure that the WTs do not cluster together, and have no bias in their locations.

We get a WT for each appearance of $a_i^{\pm 1}$ inside $\varphi(a_i)$.

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If $\varphi(a_i)$ is a long random word, we do indeed expect the letters $a_i^{\pm 1}$ to be "uniformly distributed" inside this word.

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If $\varphi(a_i)$ is a long random word, we do indeed expect the letters $a_i^{\pm 1}$ to be "uniformly distributed" inside this word.

In particular, if w is a random word of length p, we expect roughly $\frac{p}{n}$ occurences of $a_i^{\pm 1}$, distributed evenly throughout the word.

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Choosing w at random, we expect $\#_{a_i}(w) = \frac{1}{n}|w|$.

Here is the result:

Lemma

Let S_ε be the set of words w with

$$\#_{a_i}(w) > \frac{1-\varepsilon}{n}|w|.$$

for each i. Then $D(S_\varepsilon) = 1$ for any $\varepsilon > 0$.

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When $|w| = p$, the word has k occurrences of $a_i^{\pm 1}$ with probability:

$$\binom{p}{k} \left(\frac{1}{n}\right)^k \left(\frac{n-1}{n}\right)^{p-k}$$

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The probability that w has less than or equal to k occurences is

$$\sum_{j=0}^{k}\binom{p}{j}\left(\frac{1}{n}\right)^j\left(\frac{n-1}{n}\right)^{p-j}$$

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Theorem from probability: when $k < \frac{p}{n}$, we have an exponential bound

$$\sum_{j=0}^{k} \binom{p}{j} \left(\frac{1}{n}\right)^{j} \left(\frac{n-1}{n}\right)^{p-j} \leq \exp(-Ck)$$

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So:

$$\mathrm{Prob}(\#_{a_i}(w) \leq k) \leq \exp(-Ck)$$

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So

$$\#_{a_i}(w) > \frac{1-\varepsilon}{n}|w|$$

is probability 1 as $|w| \to \infty$.

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Lemma

Let S_ε be the set of words w with

$$\#_{a_i}(v) > \frac{1-\varepsilon}{n}|v|.$$

for each i, <u>for any subword</u> v with $|v| > \frac{1}{2}|w|$. Then $D(S_\varepsilon) = 1$ for any $\varepsilon > 0$.

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Let S_ε be the set of words w with

$$\frac{1-\varepsilon}{n}|v| < \#_{a_i}(v) < \frac{1+\varepsilon}{n}|v|.$$

for each i, for any subword v with $|v| > \frac{1}{2}|w|$. Then $D(S_\varepsilon) = 1$ for any $\varepsilon > 0$.

Put it all together

Our result:

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$$\geq \liminf_{p \to \infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} \frac{\#\text{WTs in remnant}}{\#WTs}$$

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$$\geq \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} \frac{\frac{1+\varepsilon}{n}|\operatorname{Rem}_\varphi a_i|}{\frac{1-\varepsilon}{n}|\varphi(a_i)|}$$

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\underline{\mathcal{E}}(N/MF) &\geq \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} \frac{\#_{a_i}(\operatorname{Rem}_\varphi a_i)}{\#_{a_i}(\varphi(a_i))} \\
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&= \frac{1+\varepsilon}{1-\varepsilon} \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} \frac{|\operatorname{Rem}_\varphi a_i|}{|\varphi(a_i)|}
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&\geq \frac{1+\varepsilon}{1-\varepsilon} \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} r
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&= \frac{1+\varepsilon}{1-\varepsilon} r \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} 1
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&\geq \frac{1+\varepsilon}{1-\varepsilon} \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} r \\
&= \frac{1+\varepsilon}{1-\varepsilon} r \liminf_{p\to\infty} \frac{1}{\#(R_r \cap S_\varepsilon)} \sum_{\varphi \in R_r \cap S_\varepsilon} 1 \\
&= \frac{1+\varepsilon}{1-\varepsilon} r
\end{aligned}$$

So we have

$$\underline{\mathcal{E}}(N/MF) \geq \frac{1+\varepsilon}{1-\varepsilon} r$$

for <u>any</u> $\varepsilon > 0$ and $r < 1$.

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But the absolute difference between these may still be fairly large.

So $MF(f) - N(f)$ can still be quite large, even when the ratio is small.

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This will require fancier counting arguments to prove.

That's all!